CASH DEMAND, LIQUIDATION COSTS AND CAPITAL MARKET EQUILIBRIUM UNDER UNCERTAINTY*

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In this paper, the portfolio and the liquidity planning problems are unified and analyzed in one model. Stochastic cash demands have a significant impact on both the composition of an individual's optimal portfolio and the pricing of capital assets in market equilibrium. The derived capital asset pricing model with cash demands and liquidation costs shows that both the market price of risk and the systematic risk of an asset are affected by the aggregate cash demands and liquidity risk. The modified model does not require that all investors hold an identical risky portfolio as implied by the Sharpe-Lintner-Mossin model. Furthermore, it provides a possible explanation for the noted discrepancies between the empirical evidence and the prediction of the traditional capital asset pricing model.

1. Introduction

The theory of portfolio demand for money and the theory of liquidity demand for money have traditionally been analyzed separately. For example, Tobin (1958) and Hicks (1962), in their applications of the Markowitz portfolio selection model to analyze the portfolio demand for money, have not considered the stochastic cash demands. On the other hand, Miller and Orr (1966), Tsai (1969), and Eppen and Fama (1971), in their applications of the stochastic inventory theory to study the liquidity demand for money, have not considered the aspect of diversification in reducing portfolio risk. The importance of unifying the portfolio demand and the liquidity demand for money in an attempt to explain an economic unit's motives for and changes in holding money is apparent, since cash or its equivalent provides a dual service of reducing the total portfolio risk and reducing the liquidation cost in meeting cash demands.

In the recent works by Chen, Jen and Zionts (1972 and 1974), it has been

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shown that individual investor's optimal portfolio decision considering stochastic cash demands is different from that without such a consideration. Thus, one will expect that the equilibrium risk-return relationships in the capital market are different with or without the consideration of cash demands and liquidation costs.

Based upon the Markowitz-Tobin portfolio selection theory, an equilibrium capital asset pricing model has been developed by Sharpe (1964), Lintner (1965) and Mossin (1966) - hereafter referred to as the SLM model. This model indicates a linear risk-return relationship for every risky asset at equilibrium. Among many important applications, the SLM model has been applied to portfolio management decisions and to devise measures of portfolio performance. However, without considering the decision-maker's cash demands, these applications are clearly inadequate. For example, the investment decisions by managers of the open-end mutual funds must include the consideration of stochastic cash demands created by share redemption, whereas that of the closed-end mutual funds do not. Thus, open-end and closed-end mutual fund managers are faced with different cash demands. As a result, the SLM model should not be applied equally to the management decisions and the performance evaluations of both open-end and closed-end mutual funds.

The purpose of this paper is to explicitly introduce a stochastic cash demand into the portfolio decisions and to derive a capital asset pricing model with cash demands and liquidation costs. In particular, we investigate the impact of cash demands and liquidation costs on the characteristics of pricing of capital assets in market equilibrium and the composition of the individual's optimal portfolio. The analysis shows that a linear risk-return relationship still exists in an equilibrium market with stochastic cash demands and liquidation costs. However, the modified model no longer possesses the property of universal separation in individuals' optimal portfolios. Furthermore, it provides a possible theoretical justification for the discrepancies between the model prediction and the empirical evidence found in Black, Jensen and Scholes (1972).

The paper is organized as follows. In section 2, we list the relevant assumptions and notation, and then derive a capital asset pricing model with cash demands and liquidation costs. Section 3 compares and contrasts the modified model with the SLM model. We focus on the differences in the market price of risk and the systematic risk between the two models. In section 4, the impact of the stochastic cash demands and liquidation costs on the individual's optimal portfolio is analyzed. The implications of our analysis are presented in the final section.

2. A capital asset pricing model with cash demands

2.1. Assumptions of the model

The following assumptions have been made in the derivation of the SLM model:
(1) All investors are risk-averse and are single-period expected utility of terminal wealth maximizers. They distinguish and select among portfolios on the basis of mean and variance of returns.

(2) Investors have homogeneous expectations with respect to the probability distributions of the future yields on risky assets.

(3) The capital market is assumed to be perfect in that:
   (a) All assets are perfectly divisible.
   (b) Information is costless and available to all market participants.
   (c) There are no taxes.
   (d) There are no transactions costs.
   (e) All investors are price takers.
   (f) All investors can borrow or lend an unlimited amount at the exogenously given riskless rate of interest. There are no restrictions on short sales of any assets.

Based upon the above assumptions, the SLM model specifies the following equilibrium risk–return relationship for any capital asset:

\[ E(R_k) = R_f + \lambda \cdot \text{cov}(R_k, R_m), \]

- \( E(R_k) \) = one plus the expected rate of return on the \( k \)th asset;
- \( R_f \) = one plus the risk-free rate of interest;
- \( \lambda \) = \([E(R_m) - R_f]/\text{var}(R_m)\) is the market price of risk;
- \( E(R_m) \) = one plus the expected rate of return on the market portfolio;
- \( \text{cov}(R_k, R_m) \) = the covariance between \( R_k \) and \( R_m \), called the systematic risk of the \( k \)th asset.

In the SLM model, it is also assumed that all assets are perfectly liquid. Perfect liquidity implies that all assets are marketable with no liquidation costs. Thus, perfect liquidity is a characteristic only possessed by perfectly marketable assets. Mayers (1972, 1973) has relaxed this assumption by providing for the existence of non-marketable assets and shown that a linear risk–return relationship exists in the equilibrium solution. However, the universal separation property no longer exists in the investors' optimal portfolios. In this paper, we modify this assumption by allowing capital assets to be less marketable with the existence of liquidation costs and stochastic cash demands. Thus, different capital assets have different degrees of liquidity. The degree of liquidity of an asset to an investor is measured by (a) the magnitude of liquidation costs to be incurred when liquidating that asset to meet his cash demands, and (b) the covariance between the return of that asset and the investor's stochastic cash demands.\(^1\)

\(^1\)This covariance has been termed the liquidity risk of the asset. The covariance between an asset's return and the investor's stochastic cash demands determines whether that asset is liquidity preferred, averse, or neutral to the investor. See Chen–Jen–Zionts (1974) for discussions on the concept and its implications on portfolio selection decisions.
More specifically, by incorporating the assets' liquidity characteristics into the model, we relax the SLM assumption (3d) and introduce a stochastic cash demand which is unique to each individual investor. Our assumptions are stated as modifications of that of the SLM model as follows:

(1') All investors are risk-averse and are single-period expected utility maximizers. Specifically, each investor’s preference function is in terms of the mean and variance of his ending portfolio value net of the liquidation costs incurred in meeting cash demands.

(2') Investors have homogeneous expectations with respect to the probability distributions of the future yields on risky assets and the cash demands. The stochastic cash demands among individual investors need not be identical. Distributions are assumed to be multivariate normal.

(3') The capital market is assumed to be perfect only to the extent that:
(a) All assets are perfectly divisible.
(b) Information is costless and available to all investors.
(c) There are no taxes.
(d) All investors are price takers.

(4') There is a riskfree liquid asset which has a certain rate of return. No liquidation (penalty) cost will be incurred using the riskfree liquid asset to meet the cash demands. However, a proportional penalty cost will be incurred if risky assets are used to meet the cash demands. There are no restrictions on short-selling of any asset.

2.2. Notation

The following notation will be employed in the subsequent analyses:

\( S_i \equiv (n \times 1) \text{ vector of the } i^{th} \text{ investor's investment in the } n \text{ available risky assets; } S_i = (S_{i1}, S_{i2}, \ldots, S_{in}) \text{ where } S_{ik} \text{ is the market value of the } i^{th} \text{ investor's holding of the } k^{th} \text{ risky asset. } S = \sum_i \sum_k S_{ik}, \text{ is the aggregate market value of all risky assets.} \)

\( \mu \equiv (n \times 1) \text{ vector of the expected return on risky assets; } \mu' = (E(R_1), E(R_2), \ldots, E(R_n)). \)

\( \Sigma \equiv [\sigma_{kl}] \text{, } (n \times n) \text{ covariance matrix of the rates of return on risky assets; } \sigma_{kl} = \text{cov}(R_k, R_l). \)

\( J_i \equiv \text{The } i^{th} \text{ investor's stochastic cash demands.} \)

\( f_i(J_i) \equiv \text{The probability density function of the } i^{th} \text{ investor's stochastic cash demands.} \)

\( F_i(J_i) \equiv \text{The cumulative distribution function of } J_i. \)

\( B_i \equiv \text{The market value of the } i^{th} \text{ investor's holding of the riskfree liquid asset.} \)

\(^2\text{The proportional penalty cost can be interpreted as the excess transfer cost of liquidating risky assets over that of liquidating the riskfree asset.}\)
2.3. The model

In making portfolio decisions at the beginning of the period, each investor must consider his stochastic cash demands that must be met at the end of the investment period. If an investor uses any risky assets to meet his cash demands, a proportional penalty cost will be incurred. Therefore, we define the stochastic penalty cost function as:

$$\phi_i = \begin{cases} c(j_i - B_i R_f), & \text{if } j_i > B_i R_f, \\ 0, & \text{if } j_i \leq B_i R_f. \end{cases}$$

Thus, the $i$th investor’s terminal portfolio value net of penalty costs, $\bar{w}_i$, can be expressed as

$$\bar{w}_i = \sum_k S_{ik} \bar{r}_k + B_i R_f - \phi_i.$$  

The expected net ending portfolio value for the investor $i$ can be expressed in matrix notation as

$$E_i = E(\bar{w}_i) = S_i \mu + B_i R_f - E(\phi_i),$$

where $E(\phi_i)$ is the expected penalty cost of liquid asset shortage.

The variance of net ending portfolio value can be expressed as

$$V_i = \text{var}(\bar{w}_i) = S_i \Sigma S_i + V(\phi_i) - 2S_i \Sigma \phi^\top,$$

$V(\phi_i)$ = the variance of the penalty cost of liquid asset shortage;

$\Sigma^\nu = (n \times 1)$ vector of the covariances between the rate of return on risky assets and the $i$th investor’s penalty cost function, $\phi_i$;

$\Sigma^\nu = \{\text{cov}(\bar{r}_1, \phi_i), \text{cov}(\bar{r}_2, \phi_i), \ldots, \text{cov}(\bar{r}_n, \phi_i)\}$.

Note that the expected penalty cost, variance of penalty cost, and covariances between the rate of return on risky assets and an investor’s penalty cost function, all involve truncated moments of $j_i$ with $B_i R_f$ as the truncation point. The mathematical properties of all truncated moments relevant to our analysis are provided in the appendix.
The portfolio decision problem faced by each investor is to maximize his preference function, $G'(E_i, V_i)$, subject to his budget constraint. All investors are assumed to be risk-averse, which implies that $\frac{\partial G'}{\partial E_i} > 0$ and $\frac{\partial G'}{\partial V_i} < 0$. Therefore, we can formulate the portfolio decision problem as the following optimization model:

Maximize $G'(E_i, V_i)$,

subject to $W_i - S_i 1 - B_i = 0$. (6)

The Lagrangian form of the model is

$L' = G'(E_i, V_i) + \lambda_i (W_i - S_i 1 - B_i)$. (7)

Therefore, the necessary conditions for optimality for the $i$th investor are

where $F'$ is the gradient operator. For example,

$\frac{\partial L'}{\partial B_i} = \left( \frac{\partial G'}{\partial E_i} \right) \left[ R_f - \frac{\partial E_i}{\partial B_i} \right] + \left( \frac{\partial V_i}{\partial V_i} \right) \left[ \frac{\partial V_i}{\partial B_i} - 2S_i \frac{\partial \xi_i}{\partial B_i} \right] - \lambda_i = 0$ (9)

Combining (8) and (9) to eliminate $\lambda_i$ and rearranging, we obtain the $i$th investor's demand for risky assets as

$S_i = \frac{1}{2} \left( \frac{\partial V_i}{\partial E_i} \right) \xi^{-1} [\mu - R_f 1 + B_i E_i(\tilde{\phi}_i)] + \frac{1}{2} \xi^{-1} B_i V_i(\tilde{\phi}_i) 1$

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where $V$ is the gradient operator. For example,

$V_{L'1} = \left( \frac{\partial L'}{\partial S_{i1}}, \frac{\partial L'}{\partial S_{i2}}, \ldots, \frac{\partial L'}{\partial S_{in}} \right)$.

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$V_{L'1} = \left( \frac{\partial L'}{\partial S_{i1}}, \frac{\partial L'}{\partial S_{i2}}, \ldots, \frac{\partial L'}{\partial S_{in}} \right)$.
At this juncture, it seems noteworthy to emphasize the liquidity services of an asset to the investor. Under uncertainty, we find that any asset, risky or riskfree, possesses liquidity services. However, the same asset can provide different liquidity services to different investors depending on their particular stochastic cash demands. Therefore, we define the internal liquidity of an asset as the liquidity services yielded by the asset to its holder. These services include the asset’s marginal contribution to the investor’s (a) expected penalty cost of liquid asset shortage, (b) variance of penalty cost of liquid asset shortage, and (c) liquidity risk measured by the covariance between the asset return and the cash demands. Thus, the demand for the riskfree liquid asset is in part the result of its contribution to provide the services for the internal liquidity demands inherent in an investor’s portfolio decision. For any investor, an increase in the holding of the riskfree liquid asset will reduce the expected penalty cost for liquidating risky assets and reduce the variance of the penalty cost. However, the marginal contribution of the riskfree liquid asset to the variance of the investor’s net ending portfolio value [see eq. (5)] depends on the liquidity risk (covariance) vector indigenous to each investor.

Since our purpose here is to derive the equilibrium risk-return relationships for capital assets, we are interested in the aggregate cash demands from all market participants and their impact on the pricing of each asset in the market. Hence, we define external liquidity of an asset as the aggregate liquidity services yielded by the asset to all participants in the market.

In equilibrium, eq. (10) must hold for all investors. Summing over all investors, and letting

\[ \sum_i \bar{\phi}_i = \bar{\phi} \quad \text{and} \quad \sum_i \bar{\phi}_i^\alpha = \bar{\phi}^\alpha, \]

we obtain:

\[
S = \frac{1}{2} \left( \sum_i \frac{\partial V_i}{\partial \mu_i} \right) \mathbf{\Phi}^{-1} \left[ \mu - R_f \mathbf{1} \right] + \frac{1}{2} \mathbf{\Phi}^{-1} \left[ \sum_i \frac{\partial V_i}{\partial E_i} \right] \mathbf{u}_i E(\bar{\phi}_i) \mathbf{1} \]

\[ + \frac{1}{2} \mathbf{\Phi}^{-1} \left[ \sum_i u_i V(\bar{\phi}_i) \right] \mathbf{1} + \mathbf{\Phi}^{-1} \mathbf{\phi} - \mathbf{\Phi}^{-1} \left[ \sum_i S_i u_i \mathbf{\phi}^\alpha \right] \mathbf{1}. \]  

(11)

Multiplying eq. (11) through by \$ yields

\[
\$S = \frac{1}{2} \left( \sum_i \frac{\partial V_i}{\partial \mu_i} \right) [\mu - R_f \mathbf{1}] + \frac{1}{2} \left[ \sum_i \frac{\partial V_i}{\partial E_i} \right] \mathbf{u}_i E(\bar{\phi}_i) \mathbf{1} \]

\[ + \frac{1}{2} \left[ \sum_i u_i V(\bar{\phi}_i) \right] \mathbf{1} + \mathbf{\Phi}^\alpha - \left[ \sum_i S_i u_i \mathbf{\phi}^\alpha \right] \mathbf{1}. \]  

(12)

This is in contrast to Kolm’s (1972) definition of internal liquidity under certainty. It is defined as “the liquidity services yielded by money in a cash balance to its holder himself”.

This is in contrast to Kolm’s external liquidity which is defined as the liquidity services yielded by money in a cash balance to agents other than the holder himself.
The $k$th row of eq. (12) defines the equilibrium risk-return relationship for asset $k$ as below,

$$
\sum_{h=1}^{n} S_h \text{cov}(\bar{R}_k, \bar{R}_h) = \frac{1}{2} \left( \sum_i \frac{\partial V_i}{\partial E_i} \right) [E(\bar{R}_k) - R_f] + \frac{1}{2} \left( \sum_i \frac{\partial V_i}{\partial E_i} \right) a_i E(\bar{\phi}_i) + \frac{1}{2} \left( \sum_i \frac{\partial V_i}{\partial E_i} \right) \delta a_i E(\bar{\phi}_i),
$$

By rearranging, we obtain a capital asset pricing model with cash demands (CAPMCD),

$$
E(\bar{R}_k) = \alpha + \lambda [S \text{cov}(\bar{R}_k, \bar{R}_m) - \text{cov}(\bar{R}_k, \bar{\phi})],
$$

where

$$
\alpha = R_f + \left\{ \sum_i \left[ -\frac{1}{2} \left( \frac{\partial V_i}{\partial E_i} \right) a_i E(\bar{\phi}_i) - \frac{1}{2} a_i V(\bar{\phi}_i) + \sum_{h=1}^{n} \frac{\partial \text{cov}(\bar{R}_k, \bar{\phi}_i)}{\partial B_i} \right] \right\}.
$$

Note that the first two terms in the numerator of the fraction are positive since $a_i E(\bar{\phi}_i) \leq 0$ and $a_i V(\bar{\phi}_i) \leq 0$ for all investors (see appendix for proofs); and

$$
\lambda = \left[ 1 \left( \sum_i V_i / \bar{E}_i \right) \right]^{-1}, \text{ the market price of risk.}
$$

This can be seen more clearly if the value-weighted return for all risky assets is substituted for $\bar{R}_k$ in eq. (13). Solving for $\lambda$, we obtain

$$
\lambda = \frac{E(\bar{R}_m) - \alpha}{S \text{var}(\bar{R}_m) - \text{cov}(\bar{R}_m, \bar{\phi})}.
$$

The market parameter $\alpha$ in eq. (13a) represents $R_f$ plus the risk-adjusted value of liquidity services provided by the riskfree liquid asset. These services include a reduction in expected penalty cost and variance of the stochastic penalty cost as a weighted average of all investors, and the aggregate adjustment for liquidity (covariance) risk.

The numerator in eq. (13b) represents the market premium or excess return over the riskfree return and the additional value derived from the liquidity services provided by the riskfree liquid asset. The denominator of eq. (13b) includes the variability risk of the market portfolio and the aggregate external liquidity risk imposed on the market.
The capital market equilibrium conditions of eqs. (13), (13a) and (13b) are expressed in terms of the penalty cost function. They can also be expressed in terms of the distribution of cash demands. Substituting the results obtained in the appendix into eqs. (13), (13a) and (13b), the capital asset pricing model with cash demands may be expressed alternatively as

\[
E(\bar{R}_h) = \alpha + \lambda [S \text{cov} (\bar{R}_k, \bar{R}_m) - c \text{cov} (\bar{R}_k, J)],
\]

(14)

where

\[
\alpha = R_f + c R_f \sum_i \left\{ \frac{1}{2} \frac{\partial V_i}{\partial E_i} [1 - F_i(B_i R_f)] + E(\tilde{\phi}_i) F_i(B_i R_f) \right\} - \text{cov} \left( \sum_h S_{ih} \bar{R}_h, j_i \right) f_i(B_i R_f) \left\{ \frac{1}{2} \sum_i \left( \frac{\partial V_i}{\partial E_i} \right) \right\},
\]

(14a)

\[
\lambda = \frac{E(\bar{R}_m) - \alpha}{S \text{var} (\bar{R}_m) + c \text{cov} (\bar{R}_m, J)},
\]

(14b)

where \( J = \sum_i j_i [1 - F_i(B_i R_f)] \) is the sum of all the market participant’s cash demands weighted by each investor’s probability of having a cash (liquid asset) shortage.

The terms in eq. (14a) have the same economic meanings as their counterpart in eq. (13a). More specifically, \( c R_f [1 - F_i(B_i R_f)] \) is the reduction in the investor’s expected penalty cost; \( c R_f E(\tilde{\phi}_i) F_i(B_i R_f) \) is the reduction in the variance of the investor’s stochastic penalty cost; and \( - \text{cov} \left( \sum_h S_{ih} \bar{R}_h, j_i \right) f_i(B_i R_f) \) is the adjustment term for liquidity risk. The sign of the adjustment term depends on whether the investor’s wealth (before the reduction of penalty cost) is positively or negatively correlated with his cash demands.\(^5\) However, (14a) shows more explicitly the effects of the probability distribution of cash demands and the probability of having liquid asset shortage on the market parameter \( \alpha \) and the determination of equilibrium capital asset prices.

From eq. (14b), we see more clearly the effect of the external liquidity risk on the market price of risk. Recall that \( J = \sum_i j_i [1 - F_i(B_i R_f)] \) is the total of each market participant’s cash demands weighted by his probability of having liquid asset shortage. Thus, the market price of risk is directly related with the aggregate external liquidity risk [i.e., \( \lambda \) increases as \( \text{cov} (\bar{R}_m, J) \) increases].

\(^5\)Since.

\[
\text{cov} (\tilde{\phi}_i, j_i) = \text{cov} \left( \sum_h S_{ih} \bar{R}_h + B_i R_f, j_i \right) = \text{cov} \left( \sum_h S_{ih} \bar{R}_h, j_i \right).
\]
3. Comparison with the SLM model

The modified capital asset pricing model as shown in (13) or (14) is more general than the SLM model. It is worth noting that if there were no penalty cost associated with the liquidation of risky assets (i.e., $c = 0$), eq. (14) would reduce to $E(R_k) = R_f + \lambda^* \text{cov}(R_k, R_m)$ which is precisely the SLM model. Therefore, we have extended the SLM model to include the effects of stochastic cash demands and the liquidation costs.

![Graph对比图示了SLM模型和包括现金需求的资本资产定价模型(CAPMCD)之间的对比。](image)

As can be seen in eq. (14), the systematic risk of an asset in the modified model has an additional component $[-c \text{cov}(R_k, J)]$ which relates the return of the asset to the aggregate cash demands. This component represents the asset's unique external liquidity services and thus influences its price in the equilibrium market. The sign of this component determines whether an asset is liquidity preferred [$\text{cov}(R_k, J) > 0$], liquidity neutral [$\text{cov}(R_k, J) = 0$], or liquidity averse [$\text{cov}(R_k, J) < 0$]. If an asset is liquidity averse, its total systematic risk is greater than that indicated by the SLM model by the magnitude $c | \text{cov}(R_k, J)|$. 
Therefore, an additional risk premium, $\lambda c \mid \text{cov} (\tilde{R}_s, J)$, is demanded to compensate for this additional component in the systematic risk. Analogously, a liquidity preferred asset reduces systematic risk which implies a reduction in risk premium. We should note that a liquidity neutral asset [i.e., $\text{cov} (\tilde{R}_s, J) = 0$] will be priced differently in equilibrium than that implied in the SLM model, since $R_f \neq \lambda$ and $\lambda^* \neq \lambda$, in general.

Finally, the capital asset pricing model with cash demands may provide an explanation for the empirical results of Black–Jensen–Scholes (1972). If the aggregate cash demand is such that $\text{cov} (\tilde{R}_s, J)$ is negative, then $\lambda^*$ is likely to be greater than $\lambda$. The resulting effect, as shown in fig. 1, will be a reduction in the market price of risk (slope) and an increase in the intercept. Therefore, the bias favoring low-beta portfolios and against high-beta portfolios found by Black–Jensen–Scholes (1972) may be due to the fact that the traditional capital asset pricing model has not incorporated the effects of stochastic cash demands and the liquidation costs of risky assets.

4. The individual’s optimal portfolio

The impact of cash demands and liquidation costs on an individual’s optimal portfolio may be analyzed through the demand equations for the risky assets. By rearranging eq. (10), we obtain

$$S_i = \left\{ \frac{1}{\tilde{V}_i} \left[ \frac{\partial \tilde{V}_i}{\partial E_i} \right]^{-1} \left[ \mu + 2 \left( \frac{\partial E_i}{\partial \tilde{V}_i} \right) \tilde{E}_i \right] + \left\{ \frac{1}{\tilde{V}_i} \left[ \frac{\partial \tilde{V}_i}{\partial E_i} \right]^{-1} \left[ -R_f + \frac{\partial E_i}{\partial \tilde{V}_i} \right] \right\} \right\},$$

which will insure that $\lambda^* > \lambda$. Note that Mayers’ (1972) explanation of Black–Jensen–Scholes (1972) empirical result did not include a change in intercept.
by its contribution to the reduction of the expected penalty cost and the variance of that penalty cost. In addition, there is an adjustment for portfolio liquidity risk, either positive or negative, whose magnitude depends on the holdings of the risky assets. The demand for each risky asset is modified by the covariability between that asset's returns and an individual's particular penalty cost function resulting from his particular stochastic cash demands.

Therefore, the risky portfolio is comprised of proportions of the risky asset dependent upon an individual's specific stochastic cash demands, $j_i$. Thus, the traditional separation property in an optimal portfolio no longer exists in the modified model. Furthermore, the proportion between risky asset $k$ and the total investment in risky assets ($S_{ik}/\sum_i S_{ih}$) is no longer independent of the individual’s taste and initial wealth.

In order to analyze the effect of cash demands upon individual investor's demands for risky assets, let us inspect eq. (10) more closely,
where $V_{ki}$ is the $k$th element of the inverse matrix of $I$.

The demand for the $k$th risky asset by the $i$th investor is represented by the $k$th row of eq. (10). Applying the definition of

$$\text{cov}(\bar{R}_i, \bar{\phi}_i) = \rho_{i\phi} \sigma(\bar{R}_i) \sigma(\bar{\phi}_i)$$

into (10), we obtain

$$S_{ik} = \frac{1}{2} \left( \sum_{i=1}^{n} V_{ki} \left[ E(\bar{R}_i) - R_f + B_i E(\bar{\phi}_i) \right] + \frac{1}{2} \sum_{i=1}^{n} V_{ki} B_i V(\bar{\phi}_i) \right)$$

$$+ \sum_{i=1}^{n} V_{ki} \rho_{i\phi} \sigma(\bar{R}_i) \sigma(\bar{\phi}_i) - \sum_{i=1}^{n} V_{ki}$$

$$\times \left\{ \sum_{h=1}^{n} S_{ih} \frac{\partial [\rho_{k\phi_h} \sigma(\bar{R}_h) \sigma(\bar{\phi}_i)]}{\partial B_i} \right\}.$$ (16)

Thus, the result obtained from the above equation that

$$\frac{\partial S_{ik}}{\partial \rho_{k\phi_i}} = \left\{ \frac{V_{kk} \sigma(\bar{R}_k) \sigma(\bar{\phi}_i) - S_{ik} \sigma(\bar{R}_k) B_i \sum_{i=1}^{n} V_{ki}}{1 + \frac{\partial [\rho_{k\phi_i} \sigma(\bar{R}_k) \sigma(\bar{\phi}_i)]}{\partial B_i} \sum_{i=1}^{n} V_{ki}} \right\} > 0$$

implies that, ceteris paribus, the investor's demand for a risky asset is greater the larger the correlation coefficient between the yield on the asset and the investor's penalty cost function.

Or more directly, the investor's demand for a risky asset is greater (or smaller) the more liquidity preferred (or the more liquidity averse) is the asset. This proposition can be proved as follows. Note from the appendix that $\text{cov}(\bar{R}_i, \bar{\phi}_i) = c \text{cov}(\bar{R}_i, j_i) [1 - F_i(B_i R_f)]$ and $\partial \sigma^2(\bar{\phi}_i) / \partial B_i = -2c R_f E(\bar{\phi}_i) F_i(B_i R_f)$. Substituting these into eq. (10) along with the definition of $\text{cov}(\bar{R}_i, \bar{\phi}_i) = \rho_{i\phi} \sigma(\bar{R}_i) \sigma(\bar{\phi}_i)$ yields

$$S_{ik} = \frac{1}{2} \left( \sum_{i=1}^{n} V_{ki} \left[ E(\bar{R}_i) - R_f + B_i E(\bar{\phi}_i) \right] \right)$$

$$- c R_f E(\bar{\phi}_i) F_i(B_i R_f) \sum_{i=1}^{n} V_{ki} + c [1 - F_i(B_i R_f)]$$

Provided that $V_{ij}, \Sigma_{i=1}^{n} V_{ki}$ and the demonstrator of the derivative are positive as the conditions for the obtained results.
Thus,

\[
\sum_{j=1}^{n} V_{kj} \rho_{kj} \sigma(R_i) \sigma(j) - \sum_{j=1}^{n} V_{kj} \times \left\{ \sum_{h=1}^{n} S_{ih} \left[ -cR_f f(B_i R_f) \rho_{hj} \sigma(R_h) \sigma(j) \right] \right\}.
\]  

(17)

Thus,

\[
\frac{\partial S_{ik}}{\partial \rho_{kj}} = \left\{ V_{ki} c \left[ 1 - F_i (B_i R_f) \right] \sigma(R_k) \sigma(j) + S_{ik} c R_f f_i (B_i R_f) \sigma(R_i) \right\} \times \sigma(j) \sum_{l=1}^{n} V_{kl} \left/ \left\{ 1 - c R_f f_i (B_i R_f) \text{ cov} (R_k, j) \sum_{l=1}^{n} V_{kl} \right\} \right. > 0,
\]

which establishes the proposition.

Another property of the investor's demand equation relates an asset's liquidity characteristic to its own standard deviation, \( \sigma(R_k) \). From eq. (16), we know that under the similar restricted conditions,

\[
\frac{\partial S_{ik}}{\partial \sigma(R_k)} \equiv 0,
\]

if

\[ \rho_{kj} \leq 0. \]

Thus, ceteris paribus, the investor's demand for a liquidity preferred asset (i.e., \( \rho_{kj} > 0 \)) is an increasing function of the asset's own standard deviation. On the other hand, his demand for a liquidity averse asset is a decreasing function of its own standard deviation.

5. Some implications

Our analysis has shown that stochastic cash demands and liquidation costs have significant impact not only on the composition of an investor's optimal portfolio, but also on the risk-return relationship in capital market equilibrium. Although a linear risk-return relationship still exists, we have shown that the traditional capital asset pricing model tends to overstate the market price of risk. Furthermore, the relevant systematic risk for a risky asset is not only its covariability with the market return, but should also include the external liquidity risk. The modified model no longer possesses the property of the SLM model that all investors hold an identical risky portfolio.

Our modified model has implications on the theory of financial intermediation. The traditional capital asset pricing model implies that there is no need for the existence of financial institutions. However, the modified model indicates that 'pooling' of the diverse patterns of individual investors' stochastic cash demands
by financial intermediaries can result in an external economy. Furthermore, the modified model explicitly incorporates liquidity risk indigenous to the investor, hence it can be used to analyze the difference in the opportunity sets resulting from the 'pooled' and the 'unpooled' cash demands.

Extending the modified model to analyze the social benefits provided by insurance companies merits some consideration. With the availability of insurance policies, an investor's demand for liquidity services can be reduced. Thus, the questions of how insurance protection affects an investor's risk-bearing behavior and what optimal amount of insurance to be purchased can be analyzed within the framework of capital market equilibrium.

Finally, our model provides a theoretically sound framework for the evaluation of the performance of mutual funds with different liquidity characteristics. Therefore, the empirical tests of the model and its application to measure investment performance appear to be of significant interest.

Appendix

The mean and the variance of ith investor's penalty cost for having liquid asset shortage is defined as

\[ E(\phi_i) = c \int_{-\infty}^{\infty} f(B_i R_f) d\phi_i \]

and

\[ V(\phi_i) = E(\phi_i^2) - [E(\phi_i)]^2. \]

By taking partial derivatives of \( E(\phi_i) \) and \( V(\phi_i) \) with respect to \( B_i \), and by rearranging it, we obtain

\[ b_i E(\phi_i) = \frac{\partial E(\phi_i)}{\partial B_i} = -c R_f [1 - F_i(B_i R_f)] < 0 \]

and

\[ b_i V(\phi_i) = \frac{\partial V(\phi_i)}{\partial B_i} = -2c R_f E(\phi_i) F_i(B_i R_f) < 0, \]

where \( F_i(B_i R_f) \) is the cumulative distribution function of \( \phi_i \). Note that \( F_i(B_i R_f) \) is the ith investor's probability of having enough liquid asset to meet the stochastic cash demands. From eqs. (A-1) and (A-2), it is obvious that both \( \frac{\partial E(\phi_i)}{\partial B_i} \) and \( \frac{\partial V(\phi_i)}{\partial B_i} \) are non-positive.

By using properties of bivariate normal distribution and the properties of truncated moments by Winkler-Roodman Britney (1972), we obtain

\[ \text{cov} (R_k, \phi_i) = E(R_k \phi_i) - E(R_k) E(\phi_i) \]

\[ = c \text{cov} (R_k, \phi_i) [1 - F_i(B_i R_f)]. \]
Taking the derivative of eq. (A-3) with respect to $B_i$, we obtain

$$
\partial \text{cov}(\bar{R}_k, \hat{\phi}_i)/\partial B_i = -cR_f \text{cov}(\bar{R}_k, \hat{J}_i)f_i(B_iR_f),
$$

(A-4)

where $f_i(B_iR_f)$ is the probability density function of $\hat{J}_i$ at $B_iR_f$, which is non-negative. Thus,

$$
\partial \text{cov}(\bar{R}_k, \hat{\phi}_i) = \partial \text{cov}(\bar{R}_k, \hat{J}_i)/\partial B_i \equiv 0,
$$

(A-5)

if $\text{cov}(\bar{R}_k, \hat{J}_i) \equiv 0$.

Since by definition $\hat{\Phi} = \sum_i \hat{\phi}_i$ and $\hat{J} = \sum_i \hat{J}_i[1 - F_i(B_iR_f)]$, it follows from eq. (A-3) that

$$
\text{cov}(\bar{R}_k, \hat{\phi}) = c \text{cov}(\bar{R}_k, \hat{J}).
$$

(A-6)

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