CAPITAL ASSET PRICING WITH PRICE LEVEL CHANGES

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I. Introduction

Economists and men of affairs have been interested in the trade-off between risk and return for centuries, but only recently has any significant progress been made in specifying this relationship. Sharpe [12], Lintner [7] and Mossin [9] have developed a single-period mean-variance model of capital asset pricing that explicitly incorporates risk. This model has been useful in examining various economic questions, including the social discount rate, the theory of capital structure, and capital budgeting.¹

Most of the major assumptions that are the basis of the capital asset pricing model do not conform to what we observe in the real world. As a result, several authors have investigated the effect of relaxing some of the basic assumptions of this model. Their analysis suggests that the basic form of the capital pricing model is robust with regard to its underlying assumptions.²

Unfortunately, little attention has been given to how the model changes when the assumption of a constant general-price-level is relaxed. Roll [10] has incorporated price level changes in a capital asset pricing model while retaining the assumption that a real risk-free interest rate exists. Since one of the major problems associated with price-level changes is that they make nominally riskless assets risky in real terms, the purpose of this paper is to present a capital asset pricing model and to illustrate difficulties involved in testing it empirically when price levels change.³

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¹For example, see Bailey and Jensen [1] and Rubinstein [11].

²See Jensen [6] for a good summary of these studies.

³After our study was almost completed we learned that Chen and Boness [4] and Long [8] have analyzed this problem but in ways that are quite different from ours.
II. Market Equilibrium in Real Terms

If price-level changes make all assets risky, it would seem that Black's [2] capital asset pricing model which assumes no riskless borrowing or lending could be used to analyze the impact of price-level changes. In the appendix we show that this is the case, but for expository purposes we will develop the model by beginning with the basic investor optimization problem. To do this we assume that:

1. All investors are risk averse single-period maximizers of expected utility of real terminal wealth with quadratic utility functions.

2. All market participants have homogeneous expectations about the expected return and variance of return for each capital asset as well as the expected value and variance of future changes in the general price level.

3. The capital markets are perfect in the sense that all assets are perfectly liquid, all investors are price-takers, there are no transactions costs or taxes, and all information can be acquired by all investors at a zero cost.

4. The relative prices of all consumption goods are constant.

5. The supply of all assets is fixed.

6. Borrowing and lending is riskless in nominal but not in real terms.

7. Direct exchanges of consumption goods result in transaction costs. These costs are eliminated by a central authority which issues token money at the beginning of the trading period and redeems it at the end of the period.4

This last assumption appears artificial and therefore requires some explanation. Typical models of a timeless exchange economy assume that one commodity is selected as the numeraire. In a one-period model such an assumption results in changes in relative commodity prices, which is a subject we believe should be studied separately in the context of the capital asset pricing model. The alternative of including money as an asset would require the inclusion of the dynamic considerations which cause individuals to hold money in a static model. What assumption 7 does, in essence, is to introduce into the model only the transactions demand for money.

In this world individuals want to maximize the expected utility of real terminal wealth subject to the constraint imposed on them by their initial:

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4 This trading period occurs before the capital markets clear since the standard capital asset pricing model takes consumption as given. Naturally all capital assets are denominated in token money. The supply of token money to be issued by the central authority is, of course, a random variable, which creates price level changes.
The nominal terminal wealth of the \( i \)th individual can be expressed as:

\[
W_i^N = \sum_{j} S_{ij} R_j^N + B_i R_f^N
\]

where:
- \( W_i^N \) is the terminal nominal wealth of the \( i \)th investor which is a random variable as denoted by the tilde,
- \( S_{ij} \) is investor \( i \)'s dollar holdings of asset \( j \) in terms of current dollars,
- \( B_i \) is the dollar amount the investor has borrowed or lent,
- \( R_i^N \) is one plus the fixed nominal rate of return on investor borrowing or lending, and
- \( R_f^N \) is one plus the nominal rate of return on the \( j \)th asset.

Equation (1) can be expressed in more relevant real terms by defining \( a \), the price level adjustment factor, as one divided by one plus the percentage change in the general price-level and writing:

\[
W_i = a W_i^N = \sum_{j} a S_{ij} R_j + B_i R_f
\]

where:
- \( R_j = a R_j^N \) which is the real return on the \( j \)th asset, and
- \( R_f = a R_f^N \) which is the real return on individual borrowing and lending.

The investor's portfolio selection problem can be expressed as a simple Lagrangian constrained optimization problem of the form:

\[
\text{Max } E[u(W_i)] + L_i (W_i - \sum_{j} S_{ij} R_j - B_i)
\]

where \( u(\cdot) \) represents the utility of wealth function. Taking partial derivatives, denoted by primes, with respect to the decision variables, i.e., the amount to hold of each capital asset and the amount to borrow or lend, gives the following first-order conditions:

\[
\begin{align*}
\text{(3a)} & \quad E[u_i' \cdot (R_j')] - L_i = 0, \quad j = 1, \ldots, N \\
\text{(3b)} & \quad E[u_i' \cdot (R_f')] - L_i = 0.
\end{align*}
\]

These conditions imply that:

\[
\begin{align*}
\text{(4)} & \quad E[u_i' \cdot (R_j')] - E[u_i' \cdot (R_f')] = \text{Cov}[u_i', (R_j' - R_f')]
\end{align*}
\]

\[
+ E(u_i') [E(R_j') - E(R_f')] = 0 \quad j = 1, \ldots, N
\]
Since the utility function is assumed to show risk aversion and to be quadratic in the form:

\[ u_i(W_i) = W_i - a_i W_i^2. \]

this implies that:

\[ u'_i = 1 - 2a_i W_i. \]

Substituting (5) into (4) and rearranging gives:

\[ \frac{\text{E}(u'_i)}{2a_i} = \text{Cov}(\tilde{W}_i, \tilde{R}_j - \tilde{R}_f) \]

as a condition of personal equilibrium.

Of primary concern, however, are the market equilibrium conditions. Since all investors have the same expectations, equation (6) can be summed over all investors to obtain:

\[ \text{E}(\tilde{R}_j) = \text{E}(\tilde{R}_f) + \frac{1}{\Sigma \text{E}(u'_i)} \left[ \text{Cov}(\tilde{W}_i, \tilde{R}_j - \tilde{R}_f) \right]. \]

Equation (2) can be summed over all investors which results in:

\[ \tilde{W}_i = \tilde{R}_m \tilde{S}_j + \tilde{R}_f \tilde{S}_i = \tilde{R}_m \Sigma \tilde{S}_j \]

since

\[ \Sigma \tilde{S}_j \tilde{R}_m = \frac{1}{\Sigma \tilde{S}_j} \]

which is the value-weighted return on all risky assets, and

\[ \tilde{EB}_i = 0 \] since economy-wide borrowing and lending must net to zero, and

\[ \tilde{S}_j = \Sigma \tilde{S}_j \]

which is the total initial market value of the \( j \)th asset.

Substituting equation (8) into (7) yields:

\[ \text{E}(\tilde{R}_j) = \text{E}(\tilde{R}_f) + \lambda \text{Cov}(\tilde{R}_m, \tilde{R}_j - \tilde{R}_f) \]

where

\[ \lambda = \frac{\Sigma \tilde{S}_j}{\Sigma \text{E}(u'_i)} \frac{1}{2a_i} \]

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Now equation (9) must hold for the expected return on the market portfolio, \( R_m \), as well as the expected return on individual risky assets in equilibrium and as a result:

\[
\lambda = \frac{E(R_m) - E(R_f)}{\text{Cov}(R_m, R_m - R_f)}.
\]

Thus equation (9) can be rewritten as:

\[
E(R_j) = E(R_f) + \frac{\text{Cov}(R_j, R_m)}{\text{Var}(R_m)} [E(R_m) - E(R_f)]
\]

which is the equilibrium relationship between risk and real return when changes in the general price level are introduced into the capital asset pricing model.

Equation (10) shows that when general price level movements are explicitly incorporated into an analysis of the risk-return relationship, the expected real return on any asset is still a linear function of its risk. What is different in this model, however, is the measure of risk. Instead of risk being measured as the ratio of the covariance of the return on the asset with the market return divided by the variance of the market return, risk is now measured by the ratio of two covariances. The difference between equation (10) and the standard capital asset pricing model is due to the assumption that the real return on the borrowing and lending rate can vary stochastically. It is easy to show that if the borrowing and lending rate is constant in real terms, then equation (10) collapses into the standard capital asset pricing model:

\[
E(R_j) = R_f + \frac{\text{Cov}(R_j, R_m)}{\text{Var}(R_m)} [E(R_m) - E(R_f)]
\]

since

\[
\frac{\text{Cov}(\tilde{R}_j - R_f, R_m)}{\text{Cov}(R_m - R_f, R_m)} = \frac{\text{Cov}(\tilde{R}_j, R_m)}{\text{Var}(R_m)}
\]

if \( R_f \) is a constant.

One interesting implication of equation (10) is that the expected real return on any risky asset is not affected by the relationship between changes in the general price level and the asset itself. This can be best seen by writing equation (10) as:

\[
E(R_j) = E(R_f) + \frac{\text{Cov}(R_j, R_m) - \text{Cov}(R_j, R_f)}{\text{Var}(R_m) - \text{Cov}(R_m, R_f)} (E(R_m) - E(R_f)).
\]
What equation (12) illustrates is that none of the terms in the equation which are unique to the $j^{th}$ asset are affected by price-level movements. The influence of price-level changes is incorporated solely on the basis of how they affect the return on the market portfolio. The intuitive reason for this is that, to the extent price-level changes are unrelated to real market returns, price-level changes are unsystematic risks which can be eliminated by diversification.

To illustrate this point, consider the case in which real market returns are independent of price-level changes. This could occur if the price-level gains and losses of individuals and firms cancel out when averaged over the entire economy. In this case the measure of risk reduces to:

$$\frac{\text{Cov}(\tilde{R}_j - \tilde{R}_f, \tilde{R}_m)}{\text{Cov}(\tilde{R}_m - \tilde{R}_f, \tilde{R}_m)} = \frac{\text{Cov}(\tilde{R}_j, \tilde{R}_m)}{\text{Var}(\tilde{R}_m)}$$

since

$$\text{Cov}(\tilde{R}_m, \tilde{R}_f) = \alpha \text{Cov}(\tilde{R}_m, \tilde{R}_f) = 0.$$

Thus, under the assumption that real market returns are not correlated with price-level changes, the equilibrium condition expressed in equation (10) is identical to that of the traditional capital asset pricing model, i.e., equation (11), except that the expected real return on borrowing and lending replaces the constant risk-free rate. The reason for this is that if price-level changes are uncorrelated with market returns, investors can eliminate the risks associated with price-level movements by simply holding the market portfolio. In sum, then, the essential features of the standard capital asset pricing model still hold even where the general price level is allowed to be a random variable if these movements are uncorrelated with the real returns on the market.

III. Market Equilibrium in Nominal Terms

Since nominal and not real returns are observed in the world, it is useful to translate equation (10) into nominal returns. This could be easily done in a formal sense by recalling that

$$\tilde{R}_j = \tilde{R}_j^N$$
$$\tilde{R}_f = \tilde{R}_f^N$$

and rewriting equation (10) as
Unfortunately, this equation involves multiplicative random variables and therefore would be difficult to test empirically.

Equation (13) can be simplified by using the proofs developed by Stevens [13] assuming again that the real return on the market portfolio is not correlated with price-level changes and that \( \tilde{\alpha}, \tilde{R}_j \) and \( \tilde{R}_m \) all have normal marginal distributions. To do this, first consider the risk term which is the ratio of two covariances. If the real market return is uncorrelated with price-level changes, then the risk measure term in real terms is the so-called beta coefficient as discussed above. This risk measure can be written as:

\[
\frac{\text{Cov}(\tilde{R}_j, \tilde{R}_m)}{\text{Var}(\tilde{R}_m)} = \frac{\text{Cov}(\tilde{\alpha} \tilde{R}_j, \tilde{R}_m)}{\text{Cov}(\tilde{\alpha} \tilde{R}_m, \tilde{R}_m)}.
\]

Using Steven's [13] equation (9) this can be expanded to

\[
(14a) \quad \frac{\text{E}(\tilde{\alpha}) \text{Cov}(\tilde{R}_j, \tilde{\alpha} \tilde{R}_m)}{\text{E}(\tilde{\alpha}) \text{Cov}(\tilde{R}_m, \tilde{\alpha} \tilde{R}_m)}
\]

and again using Steven's [13] results this becomes

\[
(14b) \quad \frac{\text{Cov}(\tilde{R}_j, \tilde{R}_m)}{\text{Var}(\tilde{R}_m)} = \frac{\text{E}(\tilde{\alpha}) \text{Cov}(\tilde{R}_j, \tilde{R}_m) + \text{E}(\tilde{R}_m) \text{Cov}(\tilde{\alpha}, \tilde{R}_m)}{\text{E}(\tilde{\alpha}) \text{Var}(\tilde{R}_m) + \text{E}(\tilde{R}_m) \text{Cov}(\tilde{\alpha}, \tilde{R}_m)}
\]

which is the risk of the jth capital asset expressed in nominal returns, and will be referred to as \( B^N_j \).

This allows us to write equation (10) as:

\[
(15) \quad \text{E}(\tilde{\alpha} \tilde{R}_j) = R^N \text{E}(\tilde{\alpha}) + [\text{E}(\tilde{\alpha} \tilde{R}_j) - R^N \text{E}(\tilde{\alpha})] B^N_j.
\]

Equation (15) still contains multiplicative random variables, but these can be eliminated by substituting the definition of the expected value of the product of two random variables and writing:\(^5\)

\[\text{That is, } \text{E}(X \cdot Y) = \text{Cov}(X, Y) + \text{E}(X) \cdot \text{E}(Y)\]
which is the market equilibrium condition in nominal terms.

Even though we have made the simplifying assumption that the return on the market portfolio is uncorrelated with price-level changes, the resulting asset-pricing model in nominal terms is not in the same form as the model in real terms. This has obvious implications for empirical testing of this model. Most empirical studies have tested the capital asset-pricing model expressed in ex-ante real terms by using nominal ex-post returns. Equation (16) demonstrates that this may lead to an inappropriate specification of the model.

Equation (16) may also be used to explain the empirical results of Black, Jensen, and Scholes [3]. Their evidence indicates that the measured relationship between portfolio excess returns and portfolio betas has a positive intercept and a slope which is less than \( E(R^m) - R^f \). However, according to the traditional model without price-level changes, the intercept should be zero and the slope should be equal to \( E(R^m) - R^f \). One possible explanation is that the capital asset-pricing model with price-level changes suggests risk should be measured by the beta coefficient expressed in real terms (or equation (14b) if the real market return is uncorrelated with price-level changes) while Black, Jensen, and Scholes [3] used nominal returns to estimate beta. Fama and McBeth [5] have provided some evidence suggesting that beta coefficients measured in nominal and real terms are approximately equal. Therefore we will assume that the two are equal in the following analysis.

Consider first the term \( \text{Cov}(\alpha, R^m) \) in equation (16). If the real return of a portfolio is uncorrelated with \( \alpha \), then this covariance must be negative. If realized returns are on average equal to their expectations, this negative covariance will be reflected in a positive intercept in the regression,

\[
(17) \quad \bar{R}_p - R^f = \alpha_0 + \alpha_1 \hat{\beta}_p + \hat{\epsilon}_p,
\]

since \( \alpha_0 \) will equal \(-\frac{\text{Cov}(\alpha, R^m)}{E(\alpha)}\). The term \( \text{Cov}(\alpha, R^m) \) will also be negative and the estimated slope coefficient will equal:

\[
(18) \quad \alpha_1 = \frac{\text{Cov}(\alpha, R^m)}{E(\alpha)} + E(R^m) - R_f
\]

which is less than the risk premium on the market portfolio in the traditional
model without price-level changes. Thus, equation (16) is consistent with the Black, Jensen, and Scholes [3] overall results.  

IV. Summary and Conclusion

A capital asset-pricing model which relates risk and return under conditions of changing price levels has been developed in this paper. The resulting model implies that price-level changes do not affect the expected real returns on individual assets except through their impact on the return of the market portfolio. If real market returns are independent of price-level movements, the model is very much like the standard capital asset-pricing model expressed in real returns. This version of the capital asset-pricing model does not, however, resolve all the difficulties associated with changing price levels, since we have assumed that the nominal default-free rate is determined outside the model and that relative prices do not change. These limitations, however, also apply to all other single-period capital asset-pricing models.

In addition, the model was converted into nominal returns by assuming that price-level changes and the real market returns are uncorrelated. The resulting equation illustrates the difficulty involved in using nominal returns to test a model expressed in real returns. The same equation also provides a possible explanation for the noted discrepancies between the empirical evidence found by Black, Jensen, and Scholes [3] and the prediction of the traditional capital asset-pricing model.

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6 Our model is not consistent with the negative intercept Black, Jensen, and Scholes [3] estimated in one of their subperiods.
REFERENCES


APPENDIX

Proof of Equation (10) Using Black's No Riskless Borrowing or Lending Capital Asset Pricing Model

Black [2] has shown that in a world with no riskless borrowing or lending the expected real return on any asset will equal:

\[(A-1)\quad \tilde{E}(R_j) = \tilde{E}(R_Z) + \frac{\text{Cov}(\tilde{R}_j, \tilde{R}_m)}{\text{Var}(R_m)} [\tilde{E}(R_m) - \tilde{E}(R_Z)]\]

where

\[\tilde{E}(R_Z) = \text{the expected real return on the minimum variance portfolio which is uncorrelated to the return on the market portfolio.}\]

This relationship must also hold for the expected real return on the default-free borrowing or lending and by subtraction we can obtain:

\[(A-2)\quad \tilde{E}(R_j) - \tilde{E}(R_f) = \frac{\text{Cov}(\tilde{R}_j, \tilde{R}_m) - \text{Cov}(\tilde{R}_f, \tilde{R}_m)}{\text{Var}(R_m)} [\tilde{E}(R_m) - \tilde{E}(R_Z)].\]

This equation can be rewritten as:

\[(A-3)\quad \tilde{E}(R_j) = \tilde{E}(R_f) + \frac{\text{Cov}(\tilde{R}_j, \tilde{R}_m) - \text{Cov}(\tilde{R}_f, \tilde{R}_m)}{\text{Cov}(R_m - \tilde{R}_f, \tilde{R}_m)} [1 - \frac{\text{Cov}(\tilde{R}_f, \tilde{R}_m)}{\text{Var}(R_m)}]\]

\[= \tilde{E}(R_f) + \frac{\text{Cov}(\tilde{R}_j - \tilde{R}_f, \tilde{R}_m)}{\text{Cov}(R_m - \tilde{R}_f, \tilde{R}_m)} [\tilde{E}(R_m) - \tilde{E}(R_Z)].\]

We know, however, that

\[(A-4)\quad \tilde{E}(R_f) - \tilde{E}(R_Z) = \frac{\text{Cov}(\tilde{R}_f, \tilde{R}_m)}{\text{Var}(R_m)} [\tilde{E}(R_m) - \tilde{E}(R_Z)]\]

and by substituting equation (A-4) into (A-3) we obtain:

\[(A-5)\quad \tilde{E}(R_j) = \tilde{E}(R_f) + \frac{\text{Cov}(\tilde{R}_j - \tilde{R}_f, \tilde{R}_m)}{\text{Cov}(R_m - \tilde{R}_f, \tilde{R}_m)} [\tilde{E}(R_m) - \tilde{E}(R_f)]\]

Which is the same as equation (10) in the text.