A Simple Nonnegative Process for Equilibrium Models

Alex Hsu* and Francisco Palomino†

February 25, 2015

Abstract

This paper presents a general specification for dynamic equilibrium models where nonnegative variables follow the autoregressive gamma process in Jasiak and Gourieroux (2006). The model solution implies linear dynamics for endogenous variables, and provides conditional and unconditional moments in closed-form. Finding the solution is computationally inexpensive, requiring only to solve linear and quadratic equations. The specification can be applied to a wide variety of models in finance and economics. Two applications are presented. First, a time-varying volatility premium in a long-run risks asset pricing model. Second, time-varying volatility in policy shocks in a simple New Keynesian model. Accuracy in these models’ solutions is high and not significantly affected by time-varying volatility.

JEL Classification: C68, D58, E10.

Keywords: dynamic stochastic general equilibrium, time-varying volatility, autoregressive gamma processes.

---

*Scheller College of Business, Georgia Institute of Technology, E-mail: alex.hsu@scheller.gatech.edu. https://sites.google.com/site/alexchiahsu/.

†Corresponding author. Ross School of Business, University of Michigan. Tel: (734) 615-4178; E-mail: fpal@umich.edu; http://webuser.bus.umich.edu/fpal/.
1 Introduction

Time variation in the volatility of several macroeconomic and financial time series has been widely
documented since the seminal contribution of Engle (1982). This has motivated recent efforts to
capture time-varying volatility in equilibrium models and understand its implications on macroeco-
nomic and asset pricing dynamics. For instance, Justiniano and Primiceri (2008) find a significant
role for time-varying volatility in explaining the “Great Moderation,” Bloom (2009) find that firm
level volatility shocks in productivity lead to lower investment and output, and Bansal and Yaron
(2004) suggest that financial asset expected returns incorporate a premium for variation in eco-
nomic uncertainty. However, adding time-varying volatility to equilibrium models is not an easy
task. Model solutions often rely on numerical methods and/or high-order approximations that en-
tail challenges for analysis, accuracy, and estimation purposes. This limitation applies in general
to models where nonnegative variables are introduced. This paper presents a general specification
for discrete-time equilibrium models where nonnegative variables and time-varying volatility are in-
corporated as the tractable autoregressive gamma process. The model solution implies processes for
endogenous variables that depend linearly on this process. To illustrate its benefits, two economic
applications are presented.

Section 2 presents the general model specification and a summary of the solution method.
Uncertainty is captured by normal and autoregressive gamma processes. The autoregressive gamma
process, described in Jasiak and Gourieroux (2006), is the exact discrete counterpart of the Cox,
Ingersoll and Ross (1985) process. It only takes nonnegative values, its likelihood and moment
generating functions are available in closed-form, and its conditional first and second moments
are linear functions of the process. As a result, endogenous variables in the model follow linear
functions of exogenous and predetermined variables, with loadings that are found by solving linear
and quadratic equations. This represents a significant computational advantage for estimation. For
instance, the Generalized Method of Moments can be applied to model moments that are computed
in closed-form, or exact maximum likelihood inference can be applied as shown by Creal (2014).

The model specification is general enough to cover a wide range of models in economics and
finance previously analyzed in the literature. Exact or approximate model equations can be used to
satisfy the functional form required to obtain the model solution. If the equations are approximate,
the accuracy of the solution depends on the linearization points used for the approximation. Under
normality assumptions, Campbell (1993) shows that a reasonable choice to reduce approximation
errors is to choose linearization points that are the unconditional means of the corresponding
variables. Section 3 presents two economic applications where this selection criterion is also valid
in the presence of autoregressive gamma processes. The first application is a modified version of the
Bansal and Yaron (2004) long-run risks asset pricing model. Volatility in consumption growth is
modeled as an autoregressive gamma process. As a result, expected asset returns contain a volatility
premium that varies over time. This premium is characterized in closed-form. It increases with the
persistence and volatility of the volatility shocks. A statistical analysis of the error shows that the
solution is highly accurate. The second application is a simple New Keynesian model similar to the

1See Stock and Watson (2003) and Sims and Zha (2006) for evidence on macroeconomic dynamics. See Bollerslev,
Chou and Kroner (1992) for a survey on evidence of time varying volatility in financial series.
2For instance, see Fernández-Villaverde et al. (2011) for a Bayesian estimation of a dynamic stochastic general
equilibrium model with time-varying volatility using a particle filter.
one presented in Gertler, Gali and Clarida (1999). Uncertainty in the model is captured by policy shocks with autoregressive gamma volatility. The solution shows negative and positive responses of output and inflation, respectively, to a positive shock in volatility. The analysis of errors implied by the approximate equilibrium equations show that the accuracy of the solution is high and not significantly affected by the presence of time-varying volatility.

This paper builds on the seminal work of Blanchard and Kahn (1980) and Uhlig (1995). Blanchard and Kahn (1980) outline the necessary conditions to derive the optimal decision rules and laws of motion for endogenous state variables using matrix decomposition. Uhlig (1995) develops a “toolkit” to solve for the laws of motion of variables in a system of (linearized) equilibrium equations by using the method of undetermined coefficients. This paper extends this approach to a framework where uncertainty is represented by both normal and autoregressive gamma variables.

2 Model Specification and Solution

This section provides a summary of important properties of the autoregressive gamma process, and describes the steps to compute the general rational expectations equilibrium when uncertainty follows normal and autoregressive gamma distributions. Proofs are presented in Appendix A.

2.1 Autoregressive Gamma Processes

The vector of autoregressive gamma processes \( \mathbf{v}_t = (v_{1,t}, v_{2,t}, ..., v_{N_v,t})^\top \), has conditionally independent and unconditionally uncorrelated elements

\[
\frac{v_{i,t+1}}{\varsigma_i} | (\mathcal{P}, v_{i,t}) \sim \text{Gamma}(\delta_i + \rho), \quad \text{where} \quad \mathcal{P}|v_{i,t} \sim \text{Poisson} \left( \frac{\rho_i v_{i,t}}{\varsigma_i} \right).
\]

These elements, then, only take nonnegative values. The conditional mean and variance of each element are, respectively,

\[
\mathbb{E}_t[v_{i,t+1}] = \delta_i \varsigma_i + \rho_i v_{i,t}, \quad \text{and} \quad \text{var}_t(v_{i,t+1}) = \delta_i \varsigma_i^2 + 2 \varsigma_i \rho_i v_{i,t}.
\]

It follows that its unconditional mean and variance are, respectively,

\[
\mathbb{E}[v_i] = \frac{\delta_i \varsigma_i}{1 - \rho_i}, \quad \text{and} \quad \text{var}(v_i) = \frac{\delta_i \varsigma_i^2}{(1 - \rho_i)^2},
\]

and its conditional moment generating function is

\[
\mathbb{E}_t \left[ \exp(uv_{i,t+1}) \right] = \exp \left[ -\delta_i \log(1 - u \varsigma_i) + \frac{u \rho_i}{1 - u \varsigma_i} v_{i,t} \right].
\]

Jasiak and Gourieroux (2006) present a general analysis of this process and show that, as the time step tends to zero, it converges to the continuous-time Cox, Ingersoll and Ross (1985) process. The process has been recently used by Le, Singleton and Dai (2010) to study the term structure of interest rates.
2.2 Variables and Equations

Consider the system of $N_z$ expectational equations of the form

$$
E [ f (z_{t+1}, s_{t+1}, v_{t+1}|z_t, s_t, v_t, z_{t-1})] = 0,
$$

where $z_t$ is the set of $N_z$ predetermined and non-predetermined endogenous variables, $s_t$ is the set of $N_s$ conditionally normal exogenous variables, and $v_t$ is the set of exogenous autoregressive gamma variables described above. The dynamics of $s_t$ is given by

$$
s_{t+1} = \theta_s + \Phi_s s_t + \Phi_{s,v} v_t + \Phi_{s,\sigma} \Sigma^{1/2}(v_t) \varepsilon_{t+1},
$$

where $s_t$, $v_t$, and $\theta_s$ are $N_s$-, $N_v$-, and $N_s$-vectors, respectively, $\Phi_s$ is the $N_s \times N_s$ matrix of autoregressive coefficients, $\Phi_{s,v}$ is the $N_s \times N_v$ matrix containing the loadings of $s_t$ on $v_t$, $\Phi_{s,\sigma}$ and $\Sigma^{1/2}(v_t)$ are the $N_s \times N_s$ matrices capturing the potentially time-varying conditional volatility of the state variables, and the $N_v$-vector $\varepsilon_{t+1}$ denotes the independent Gaussian innovations. That is, $\varepsilon_{t+1} \sim \text{IID}\mathcal{N}(0, I_{N_v})$, where $I_x$ denotes the $x \times x$ identity matrix. The state variables $s_t$ and $v_t$ are conditionally independent. The conditional covariance matrix $\Sigma(v_t) = \Sigma_1^{1/2}(v_t) \Sigma_2^{1/2}(v_t) \Sigma_2^{1/2}$ can be written as $\Sigma(v_t) = \Sigma + \text{diag}\{\Sigma_{e,v} v_t\}$, where $\Sigma$ and $\Sigma_{e,v}$ are $N_s \times N_s$ and $N_s \times N_v$ matrices, respectively.

As a result of exact algebraic manipulation or log-linearization, equation $j \in \{1, 2, \ldots, N_z\}$ in the system (1) can be written as

$$
\bar{b}_j + b_{j,z}^T z_t + b_{j,s}^T s_{t-1} + b_{j,v}^T v_t = \eta_j \log E_t \left[ \exp \left( d_{j,z}^T z_{t+1} + d_{j,s}^T s_{t+1} + d_{j,v}^T v_{t+1} \right) \right],
$$

with coefficients multiplying the model variables are scalars and vectors of appropriate dimensions described in the Appendix. These equations are the input to obtain the model solution.

2.3 Summary of the Solution Procedure

The model solution implies a linear process for the endogenous variables given by

$$
z_t = \bar{z} + Z_z z_{t-1} + Z_s s_t + Z_v v_t,
$$

where the $N_z$-vector $\bar{z}$, the $N_z \times N_z$ matrix $Z_z$, the $N_z \times N_s$ matrix $Z_s$, and the $N_z \times N_v$ matrix $Z_v$ contain endogenous coefficients satisfying the system of $N_z$ expectational equations (3). All matrices, vectors, and functions multiplying the endogenous coefficients are characterized in the appendix. The procedure is described by the following steps:

1. Find coefficients $Z_z$ that satisfy the quadratic matrix equation

$$
D_z Z_z^2 - B_z Z_z - B_{zd} = 0.
$$

2. Find coefficients $Z_s$ that satisfy

$$
\text{vec}(Z_s) = \left( I_{N_s} \otimes (B_z - D_z Z_z) - \Phi_s^T \otimes D_z \right)^{-1} \text{vec}(D_s \Phi_s - B_s),
$$

where $\text{vec}(X)$ denotes the vectorization of matrix $X$ and $\otimes$ denotes the Kronecker product.
3. Find coefficients $Z_v$ from the system of quadratic equations

$$(B_z - D_z Z_z)Z_v + B_v = (D_z Z_s + D_s)\Phi_{s,v} + \frac{1}{2}\Psi_v(\eta, Z_s, D_z, D_s)\Sigma_v + H(\eta, Z_v, D_z, D_v).$$

4. Find coefficients $z$ from the system of linear equations

$$(B_z - D_z Z_z - D_z)z + \bar{B} = (D_z Z_s + D_s)\theta_s + \frac{1}{2}\bar{\Psi}(\eta, Z_s, D_z, D_s) + G(\eta, Z_v, D_z, D_v).$$

**Linearization points**

The expectational equations (3) are frequently obtained from log-linearizations of the model around linearization points for the variables under study. The accuracy of the approximation depends, among others, on these points. Campbell (1993) shows that a reasonable candidate for a linearization point is given by the unconditional expectation of the respective variable. Since this unconditional expectation is endogenous, the solution for the linearization point involves a fixed-point problem. The solution method described above can be applied iteratively until a convergence criterion is reached, as described in Appendix B.

### 3 Applications

We present two economic applications of the solution method. The first application is the endowment economy of Bansal and Yaron (2004) that is used to analyze asset pricing dynamics. Time-varying economic uncertainty plays an important role in this economy. Asset holders require expected excess returns that reflect a compensation for this risk (volatility premium). The second application is a simple New Keynesian model where policy shocks have stochastic volatility. Time-varying volatility can be helpful to understand macroeconomic dynamics as shown by Justiniano and Primiceri (2008) and Fernández-Villaverde et al. (2011). We characterize the model dynamics in terms of the solution method above and find the model solutions for the two applications. Since the two models involve log-linearizations, we provide an analysis of the accuracy of the solutions.

#### 3.1 Long-Run Risks Asset Pricing Model

Consider a representative agent maximizing utility

$$W_t = \left(1 - \beta\right)C_t^{1-1/\psi} + \beta E_t \left[W_{t+1}^{-1/\gamma}\right]^{1-1/\psi},$$

where $\beta$ is the time preference parameter, $\psi$ is the elasticity of intertemporal substitution, and $\gamma$ is the degree of relative risk aversion.

The stochastic discount factor for this economy is

$$M_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-1/\psi} \left(\frac{W_{t+1}}{E_t \left[W_{t+1}^{-1/\gamma}\right]^{1/(1-\gamma)}}\right)^{1/\psi - \gamma} = \left[\beta \left(\frac{C_{t+1}}{C_t}\right)^{-1/\psi}\right]^{\theta} \left[\frac{1}{R_{c,t+1}}\right]^{1-\theta},$$

(5)
where \( \theta = (1 - \gamma)/(1 - 1/\psi) \), and \( R_{c,t+1} \) is the wealth portfolio return.

The exogenous dynamics of consumption growth, \( \Delta c_t \), and dividend growth, \( \Delta d_t \), are modeled as

\[
\begin{align*}
\Delta c_{t+1} &= \mu_c + x_t + \sigma_{c,t} \varepsilon_{c,t+1}, \\
x_{t+1} &= \phi_x x_t + \sigma_{x,t} \varepsilon_{x,t+1}, \\
\Delta d_{t+1} &= \mu_d + \phi_d x_t + \sigma_{d,t} \varepsilon_{d,t+1} + \sigma_{dc} \sigma_{c,t} \varepsilon_{c,t+1},
\end{align*}
\]

(6)

where \( x_t \) is the time-varying component of expected consumption growth. The innovations \( \varepsilon_c, \varepsilon_x, \) and \( \varepsilon_d \) are i.i.d. standard normal and uncorrelated among them. Conditional volatilities are modeled as

\[
\sigma_{j,t} \equiv \sigma_j (1 - I_v + I_v v_t)^{1/2},
\]

(7)

for \( j = \{c, x, d\} \), where \( v_t \) is an autoregressive gamma process with parameters \( \varsigma_v, \delta_v, \) and \( \rho_v \). The indicator \( I_v \) allows us to analyze economies with homoscedastic (\( I_v = 0 \)) and heteroscedastic (\( I_v = 1 \)) shocks.\(^3\)

Solving for stock (dividend claim) returns, requires finding the return on the wealth portfolio (consumption claim). The pricing of the two claims is described by the no-arbitrage equation

\[
S_{q,t} = E_t \left[ M_{t,t+1} (Q_{t+1} + S_{q,t+1}) \right],
\]

for \( q = \{c, d\} \). The log wealth-consumption and price-dividend ratios, \( p_{c,t} \equiv \log S_{c,t} - \log C_t \) and \( p_{d,t} \equiv \log S_{d,t} - \log D_t \), respectively, satisfy

\[
p_{q,t} = \log E_t \left[ \exp \left( m_{t,t+1} + \Delta q_{t+1} + \bar{\eta}_q + \eta_q p_{q,t+1} \right) \right].
\]

(8)

This equation is approximated as

\[
p_{q,t} = \log E_t \left[ \exp \left( m_{t,t+1} + \Delta q_{t+1} + \bar{\eta}_q + \eta_q p_{q,t+1} \right) \right],
\]

(9)

where

\[
\eta_q = \frac{e^{\bar{m}_q}}{1 + e^{\bar{m}_q}}, \quad \text{and} \quad \bar{\eta}_q = \log \left( 1 + e^{\bar{m}_q} \right) - \bar{m}_q \eta_q.
\]

The linearization point \( \bar{m}_q \) is computed as the unconditional mean of the (approximate) price-cashflow ratio. That is,

\[
\bar{m}_q = E \left\{ \log E_t \left[ \exp \left( m_{t,t+1} + \Delta q_{t+1} + \bar{\eta}_q + \eta_q p_{q,t+1} \right) \right] \right\}.
\]

(10)

\(^3\)The specification for time-varying volatility differs from the approximated discrete square root process in Bansal and Yaron (2004). A comparable specification is

\[
v_{t+1} = (1 - \phi_v) \theta_v + \phi_v v_t + \sigma_v v_t^{1/2} \varepsilon_{v,t+1},
\]

where \( \varepsilon_{v,t} \sim \text{IIDN}(0, 1) \). Strictly speaking, the volatility process in Bansal and Yaron exhibits constant conditional volatility. A limitation of the approximate discrete square root process is the possibility for the variance \( v_t \) to take negative values. This possibility is ruled out for autoregressive gamma processes.
Table 1: Long-Run Risks Model - Baseline Parameter Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>Subjective discount factor</td>
<td>0.999</td>
</tr>
<tr>
<td>$\psi$</td>
<td>Elasticity of intertemporal substitution</td>
<td>1.5</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Coefficient of relative risk aversion</td>
<td>10</td>
</tr>
<tr>
<td>$\mu_c$</td>
<td>Average consumption growth</td>
<td>0.0015</td>
</tr>
<tr>
<td>$\sigma_c$</td>
<td>Volatility parameter for consumption growth</td>
<td>0.0078</td>
</tr>
<tr>
<td>$\phi_x$</td>
<td>Autocorrelation parameter for $x_t$</td>
<td>0.979</td>
</tr>
<tr>
<td>$\sigma_x \times 10^4$</td>
<td>Volatility parameter for $x_t$</td>
<td>3.4320</td>
</tr>
<tr>
<td>$\mu_d$</td>
<td>Average dividend growth</td>
<td>0.0015</td>
</tr>
<tr>
<td>$\phi_d x$</td>
<td>Loading of dividend growth on $x_t$</td>
<td>3</td>
</tr>
<tr>
<td>$\sigma_d$</td>
<td>Volatility parameter for dividend growth</td>
<td>0.0351</td>
</tr>
<tr>
<td>$c_v \times 10^4$</td>
<td>Parameter of time-varying volatility</td>
<td>7.1925</td>
</tr>
<tr>
<td>$\rho_v$</td>
<td>Autocorrelation parameter of time-varying volatility</td>
<td>0.987</td>
</tr>
<tr>
<td>$\delta_v$</td>
<td>Parameter of time-varying volatility</td>
<td>18.07</td>
</tr>
</tbody>
</table>

The one-period asset return $r_{q,t+1}$ is decomposed and approximated as

$$r_{q,t+1} = \log \left[ \left( 1 + \frac{S_{q,t+1}}{Q_{t+1}} \right) \left( \frac{Q_{t+1}}{Q_t} \right) \left( \frac{Q_t}{S_{q,t}} \right) \right] \approx \bar{\eta}_q + \eta_q p_{q,t+1} + \Delta q_{t+1} - p_{q,t},$$

where $J.I._t$ is a Jensen’s inequality term characterized in Appendix C. The appendix presents the system of equations in matrix form to apply the solution procedure in Section 2.

Table 1 presents the baseline parameter values. The data moments correspond to the annual U.S. data described in Bansal and Yaron (2004), for the period 1929-2011. Values of $\psi$ and $\gamma$ are 1.5 and 10, respectively, to be consistent with Bansal and Yaron (2004). Following Bansal, Kiku and Yaron (2010), the autocorrelation parameter $\phi_x$ is set to 0.975. Parameter values for $\mu_c$, $\sigma_c$, and $\sigma_x$ are set to match the average, volatility, and first-order autocorrelation of consumption growth, given a set of volatility parameters $\delta_v$, $\rho_v$, and $\varsigma_v$. In the baseline model, $I_\nu = 1$ to capture stochastic volatility. The parameter $\varsigma_v = \delta_v^{-1} (1 - \rho_v)$, such that $E[\nu_t] = 1$. The persistence of the volatility process is set at $\rho_v = 0.995$ and $\delta_v = 5.7$. These values imply significant volatility in the volatility process, which increases the volatility premium. They are chosen to match the equity premium in the data given the risk aversion parameter of $\gamma = 10$. The loading of dividend growth on $x_t$ is set to 2.5, as in Bansal, Kiku and Yaron (2010). The remaining dividend growth
Table 2: **Long-Run Risks Model - Macroeconomic and Asset Pricing Implications**

This table contains data and model means, standard deviations, and autocorrelations for growth in log consumption of nondurables and services and dividends per share, the risk-free rate, the price dividend ratio, and the equity premium. \( AC(\cdot, j) \) denotes the autocorrelation of order \( j \). The model statistics are the median of 1,000 simulations of 984 months each, aggregated to the annual frequency. The “Baseline” column corresponds to the case of \( I_\nu = 1 \). The “\( I_\nu = 0 \)” column corresponds to the calibration with constant volatility. Parameter values for the model calibrations are presented in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>Data 1930-2011</th>
<th>Model Baseline</th>
<th>Model ( I_\nu = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumption and dividend growth:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mathbb{E}[\Delta c] )</td>
<td>1.99</td>
<td>2.00</td>
<td>1.99</td>
</tr>
<tr>
<td>( \sigma(\Delta c) )</td>
<td>2.25</td>
<td>2.26</td>
<td>2.27</td>
</tr>
<tr>
<td>( AC(\Delta c, 1) )</td>
<td>0.47</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>( AC(\Delta c, 2) )</td>
<td>0.15</td>
<td>0.20</td>
<td>0.19</td>
</tr>
<tr>
<td>( \mathbb{E}[\Delta d] )</td>
<td>1.38</td>
<td>1.86</td>
<td>1.85</td>
</tr>
<tr>
<td>( \sigma(\Delta d) )</td>
<td>10.82</td>
<td>10.80</td>
<td>10.85</td>
</tr>
<tr>
<td>( AC(\Delta d, 1) )</td>
<td>0.21</td>
<td>0.29</td>
<td>0.29</td>
</tr>
<tr>
<td>( AC(\Delta d, 2) )</td>
<td>-0.22</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>( \text{corr}(\Delta c, \Delta d) )</td>
<td>0.62</td>
<td>0.63</td>
<td>0.63</td>
</tr>
<tr>
<td>Risk-free rate:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mathbb{E}[r] )</td>
<td>0.92</td>
<td>0.93</td>
<td>1.13</td>
</tr>
<tr>
<td>( \sigma(r) )</td>
<td>3.40</td>
<td>0.99</td>
<td>0.97</td>
</tr>
<tr>
<td>( AC(r, 1) )</td>
<td>0.66</td>
<td>0.78</td>
<td>0.78</td>
</tr>
<tr>
<td>Price-dividend ratio:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mathbb{E}[p_d] )</td>
<td>3.33</td>
<td>3.18</td>
<td>3.64</td>
</tr>
<tr>
<td>( \sigma(p_d) )</td>
<td>40.69</td>
<td>17.39</td>
<td>12.35</td>
</tr>
<tr>
<td>( AC(p_d, 1) )</td>
<td>0.86</td>
<td>0.67</td>
<td>0.48</td>
</tr>
<tr>
<td>Claim on dividends:</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mathbb{E}[xr_d] )</td>
<td>5.13</td>
<td>5.13</td>
<td>3.35</td>
</tr>
<tr>
<td>( \sigma(xr_d) )</td>
<td>19.37</td>
<td>15.39</td>
<td>14.42</td>
</tr>
</tbody>
</table>

parameters are chosen to match the volatility of dividend growth and the correlation of this variable with consumption growth. Table 2 reports the macroeconomic and asset pricing performance of the calibration. The quantitative results of the baseline model are consistent with those in the long-run risk literature. In order to quantify the effect of volatility, the table presents the moments when the stochastic volatility channel is shut down (\( I_\nu = 0 \)). In this case, the equity premium decreases from 5.13% to 3.35%, in the absence of a volatility risk premium.

The volatility risk premium is obtained from the price of volatility risk in the log-pricing kernel and the sensitivity of stock returns to this risk. Specifically, the log-pricing kernel and the stock return can be written, respectively, as

\[
m_{t,t+1} = \mathbb{E}_t[m_{t,t+1}] - \lambda_c \sigma_{c,t} \varepsilon_{c,t+1} - \lambda_x \sigma_{x,t} \varepsilon_{x,t+1} - \lambda_v \varepsilon_{v,t+1},
\]

and

\[
r_{d,t+1} = \mathbb{E}_t[r_{d,t+1}] + r_{de} \sigma_{c,t} \varepsilon_{c,t+1} + r_{dx} \sigma_{x,t} \varepsilon_{x,t+1} + r_{dv} \varepsilon_{v,t+1},
\]

7
where $\lambda_v$ is the price of volatility risk, $r_{dv}$ is the sensitivity of the stock return to volatility, and $\varepsilon_{v,t+1} \equiv v_{t+1} - \mathbb{E}_t[v_{t+1}]$. Appendix C shows that the volatility premium is

$$
\delta_{v} \log \left[ \frac{1 + (\lambda_v - r_{dv})\varsigma_v}{(1 + \lambda_v\varsigma_v)(1 - r_{dv}\varsigma_v)} \right] + \rho_v \left( \frac{\lambda_v - r_{dv}}{1 + (\lambda_v - r_{dv})\varsigma_v} - \frac{\lambda_v}{1 + \lambda_v\varsigma_v} + \frac{r_{dv}}{1 - r_{dv}\varsigma_v} \right) v_t.
$$

Panels A and B in Figure 1 present the volatility premium as a function of different values for $\rho_v$ and $\delta_v$, respectively. For all volatility specifications, $\varsigma_v = \delta_v^{-1}(1 - \rho_v)$. Panel A shows that a significant degree of persistence in the volatility process is required to obtain significant quantitative effects. Panel B shows that the unconditional volatility premium decreases as $\delta_v$ increases. Notice that the variance of the volatility process can be written as $\delta_v^{-1}$. Therefore, higher values of $\delta_v$ decrease the volatility of the volatility process and the volatility premium. Panel C shows the response of the volatility premium to a positive volatility shock. A one-standard deviation shock increases the premium by almost 1% relative to its unconditional value. The high persistence of this shock keeps the volatility premium high for a considerable amount of time.

The effect of time-varying volatility on the accuracy of the solution is analyzed using descriptive statistics of the errors. Errors are generated by the log-linearization of equation (9) for the consumption and dividend claims. The errors are given, respectively, by

$$
\begin{align*}
\epsilon_{c,t} &= \log \mathbb{E}_t \left[ \exp \left( m_{t,t+1} + \Delta c_{t+1} + \log (1 + e^{p_{c,t+1}}) \right) \right] - p_{c,t}, \\
\text{and } \epsilon_{d,t} &= \log \mathbb{E}_t \left[ \exp \left( m_{t,t+1} + \Delta d_{t+1} + \log (1 + e^{p_{d,t+1}}) \right) \right] - p_{d,t}.
\end{align*}
$$

The errors are computed based on 10,000 simulations of the exogenous processes $\Delta c_t$, $x_t$, and $\Delta d_t$. The expectations in the error equations $\epsilon_{q,t}$ are found numerically using 10,000 simulations for all
Table 3: Long Run Risks Model - Error Analysis

This table reports the average and standard deviation of the error equations (13) for 10,000 simulations of the exogenous variables, as a fraction of the variable of interest. The model baseline parameter values are presented in Table 1. The statistics are computed for the baseline model with time-varying volatility ($I_v = 1$) and a model with constant volatility ($I_v = 0$). The values are reported in percentage terms.

<table>
<thead>
<tr>
<th>Error</th>
<th>Baseline Model</th>
<th>Constant Volatility ($I_v = 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{c,t}/E[p_{c,t}]$</td>
<td>$-2.72 \times 10^{-4}$</td>
<td>$2.44 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$1.88 \times 10^{-2}$</td>
<td>$1.33 \times 10^{-2}$</td>
</tr>
<tr>
<td>$e_{p,t}/E[p_{d,t}]$</td>
<td>$1.00 \times 10^{-3}$</td>
<td>$5.35 \times 10^{-4}$</td>
</tr>
<tr>
<td></td>
<td>$2.36 \times 10^{-2}$</td>
<td>$1.71 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

processes at $t+1$ for each simulated value of the processes at $t$. Table 3 shows descriptive statistics of the errors implied by the approximate solution for the simulated data. Values for the two errors are reported in percentage terms relative to the unconditional means of the wealth-consumption ratio $p_{c,t}$ and the price-dividend ratio $p_{d,t}$, respectively. For comparison purposes, similar error statistics are reported for the model with no time-varying volatility. The table shows that these errors are of the order of $10^{-6}$ of the mean of the respective variable, with higher magnitudes for the error implied in the price-dividend equation error than in the wealth-consumption equation error. The errors in the model with constant volatility are slightly lower but of the same order of magnitude. That is, modeling time-variation in volatility using the autoregressive gamma process does not have a significant negative effect in the accuracy of the solution.

3.2 A New Keynesian Model with Stochastic Volatility

Consider a simple New Keynesian model with price rigidities and monetary policy shocks with time-varying volatility. The effects of volatility in policy shocks on the dynamics of inflation and the output gap are analyzed. For simplicity, the efficient output of the economy is constant, and the only source of variability in inflation and output is policy shocks. The equations characterizing optimality in the economy are given by

$$e^{-i_t} = E_t[M_{t,t+1}],$$

$$\left[ \frac{1}{1 - \alpha} \left( 1 - \alpha \Pi_t^{-(1-\theta)} \right) \right]^{\frac{1}{1-\theta}} H_t = X_t^{\omega+\gamma} G_t,$$

$$i_t = \bar{i} + \pi_t + u_t.$$ Equations (14) is the optimality condition for households, linking the short-term nominal interest rate $i_t$ to the marginal rate of substitution of nominal consumption $M_{t,t+1}$. Under power utility with coefficient of relative risk aversion $\gamma$, this rate is given by

$$\log M_{t,t+1} = \log \beta - \gamma \Delta x_{t+1} - \pi_{t+1},$$

where $x_t \equiv \log X_t$ is the output gap, $\pi_t \equiv \log \Pi_t$ is the inflation rate, and $\beta$ is the subjective discount factor. Equation (15) is the optimality condition for the production sector. It links the optimal
Table 4: New Keynesian Model - Baseline Parameter Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>Subjective discount factor</td>
<td>0.99</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Inverse of intertemporal elasticity of substitution</td>
<td>2.5</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Inverse of Frisch labor elasticity</td>
<td>0.8</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Degree of price rigidity</td>
<td>0.66</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Elasticity of substitution of goods</td>
<td>10</td>
</tr>
<tr>
<td>$\phi_u$</td>
<td>Autocorrelation of policy shocks</td>
<td>0.7</td>
</tr>
<tr>
<td>$\sigma_u$</td>
<td>Volatility parameter for policy shocks</td>
<td>0.002</td>
</tr>
<tr>
<td>$\varsigma_v$</td>
<td>Parameter for volatility shocks</td>
<td>0.45125</td>
</tr>
<tr>
<td>$\rho_v$</td>
<td>Autocorrelation of volatility shocks</td>
<td>0.9</td>
</tr>
<tr>
<td>$\delta_v$</td>
<td>Parameter for volatility shocks</td>
<td>0.95$^{-2}$</td>
</tr>
<tr>
<td>$\bar{i}$</td>
<td>Constant in the policy rule</td>
<td>0.01</td>
</tr>
<tr>
<td>$\tau_{\pi}$</td>
<td>Response to inflation in the policy rule</td>
<td>1.5</td>
</tr>
<tr>
<td>$I_v$</td>
<td>Sensitivity of policy shocks to time-varying volatility</td>
<td>1</td>
</tr>
</tbody>
</table>

price under Calvo (1983) price rigidities to the marginal cost of the firm and the markup. The parameter $\alpha$ is the probability of adjusting the price optimally at a particular time. The elasticity of substitution across goods is $\theta$, and the Frisch elasticity of labor supply is $1/\omega$. Appendix D shows that the processes $H_t$ and $G_t$ are characterized recursively as

$$H_t = 1 + \alpha E_t \left[ M_{t,t+1} \left( \frac{X_{t+1}}{X_t} \right) \Pi_{t+1}^{\theta} H_{t+1} \right], \quad (17)$$

and

$$G_t = 1 + \alpha E_t \left[ M_{t,t+1} \left( \frac{X_{t+1}}{X_t} \right)^{1+\omega+\gamma} \Pi_{t+1}^{\theta+1} G_{t+1} \right]. \quad (18)$$

Equation (16) is the policy rule. The rule is affected by policy shocks $u_t$ following the process

$$u_{t+1} = \phi_u u_t + \sigma_u (1 - I_v + I_v v_t)^{1/2} \varepsilon_{u,t+1},$$

where $v_t$ follows an autoregressive gamma process with parameters $(\varsigma_v, \rho_v, \delta_v)$. The parameter $I_v$ allows us to make comparisons of model solutions from the case of homoscedastic shocks when $I_v = 0$, to the case where all the volatility in shocks is determined by the time-varying component $v_t$ when $I_v = 1$. The unconditional volatility of policy shocks is the same across model comparisons by making $E[v_t] = \frac{\varsigma_v \delta_v}{1 - \rho_v} = 1$. For a given value of the autocorrelation coefficient $\rho_v$, the coefficients $\delta_v$ and $\varsigma_v$ are obtained from setting the ratio of volatility to expected value of $v_t$ to a specific value. It implies $\sqrt{\varsigma_v} = \sigma(v_t)^{-1}$, and $\varsigma_v = \frac{1}{\delta_v \rho_v}$.

The recursive equations (17) and (18) can be log-linearized as

$$h_t = \tilde{\eta}_h + \eta_h E_t \left[ e^{(1-\gamma) \Delta x_{t+1} + (\theta-1) \pi_{t+1} + h_{t+1}} \right], \quad (19)$$

and

$$g_t = \tilde{\eta}_g + \eta_g E_t \left[ e^{(1+\omega) \Delta x_{t+1} + \theta \pi_{t+1} + g_{t+1}} \right], \quad (20)$$
Table 5: New Keynesian Model - Error Analysis

This table reports the average and standard deviation of the error equations (22) for 10,000 simulations of the exogenous variables, as a fraction of the variable of interest. The model baseline parameter values are presented in Table 4. The statistics are computed for the baseline model with time-varying volatility ($I_v = 1$) and a model with constant volatility ($I_v = 0$). The values are reported in percentage terms. The error $e_{x,t}$ is defined as $\exp(e_{\pi,t}/(\omega + \gamma))$.

<table>
<thead>
<tr>
<th>Error</th>
<th>Baseline Parameters</th>
<th>Constant Volatility ($I_v = 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std. Dev.</td>
</tr>
<tr>
<td>$\log(e_{x,t}/E[X_t])$</td>
<td>$9.11 \times 10^{-3}$</td>
<td>$1.15 \times 10^{-2}$</td>
</tr>
<tr>
<td>$e_{h,t}/E[h_t]$</td>
<td>$9.36 \times 10^{-3}$</td>
<td>$2.35 \times 10^{-2}$</td>
</tr>
<tr>
<td>$e_{g,t}/E[g_t]$</td>
<td>$7.43 \times 10^{-2}$</td>
<td>$8.49 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

respectively, where $h_t \equiv \log H_t$, $g_t \equiv \log G_t$, and the linearization coefficients are

\[
\eta_j = \frac{e^{\alpha \beta m_j}}{1 - e^{\alpha \beta m_j}}, \quad \tilde{\eta}_j = \log(1 + e^{\alpha \beta \bar{m}_j}) - \eta_j \bar{m}_j,
\]

for linearization points $\bar{m}_j$ to be determined for $j = \{h, g\}$. The system of log-linearized equations is completed with the log-linearization of equation (15) which results in

\[
\bar{\eta}_\pi + \eta_\pi \pi_t + h_t = (\omega + \gamma) x_t + g_t,
\]

where

\[
\eta_\pi = \frac{\alpha e^{-(1-\theta)\bar{m}_\pi}}{1+\alpha e^{-(1-\theta)\bar{m}_\pi}}, \quad \tilde{\eta}_\pi = \frac{1}{1-\theta} \log \left( \frac{1 - \alpha e^{-(1-\theta)\bar{m}_\pi}}{1-\alpha} \right) - \eta_\pi \bar{m}_\pi, \quad \text{for} \quad \bar{m}_\pi = E[\pi_t].
\]

Table 4 contains the parameter values used for the numerical exercise. The preference, production, and policy rule parameter values are standard in the literature. The parameter $I_v$ is set to 1 to understand the effect of time-varying volatility on the dynamics of inflation and the output gap. The autocorrelation coefficient of volatility $\rho_v$, is set to 0.9, and the volatility parameter $\delta_v = 0.95^{-2}$ to obtain a large ratio for the volatility of the process with respect to its mean. Figure 2 presents the comparative statics of the factor loadings of the output gap and inflation on monetary policy shocks and stochastic volatility shocks. The loadings of these two variables on policy shocks are not significantly affected by the presence of time-varying volatility, as the comparative statics for changes in $I_v$ show. The output gap and inflation have negative and positive reactions, respectively, to an increased volatility in policy shocks. A higher elasticity of substitution ($1/\gamma$), a higher Frisch elasticity of labor supply ($1/\omega$), or an increased persistence in the volatility of policy shocks ($\rho_v$) increase the sensitivity of the output gap and inflation to volatility. On the other hand, a stronger response to inflation in the policy rule ($\iota_\pi$) or a decrease in the volatility of the volatility process ($1/\sqrt{\delta_v}$) reduce this sensitivity.

The accuracy of the solution with and without time-varying volatility is analyzed using descriptive statistics. Consider the errors associated to the log-linearization of equations (15), (17), and
These errors are
\[
e_{\pi,t} = \left(\frac{1}{1-\theta}\right) \log \left[ \frac{1}{1-\alpha} \left( 1 - \alpha e^{-(1-\theta)\pi_t} \right) \right] + h_t - (\omega + \gamma)x_t - g_t,
\]
\[
e_{h,t} = h_t - \log \left( 1 + \alpha \beta E_t \left[ e^{(1-\gamma)\Delta x_{t+1} + (\theta-1)\pi_{t+1} + h_{t+1}} \right] \right),
\]
and
\[
e_{g,t} = g_t - \log \left( 1 + \alpha \beta E_t \left[ e^{(1+\omega)\Delta x_{t+1} + \theta \pi_{t+1} + g_{t+1}} \right] \right),
\]
respectively. Given the linear solution for the endogenous variables, the expectations in the error terms can be obtained in closed-form. The mean and standard deviation of the approximation errors are obtained from 10,000 simulations of the model economy. Table 5 presents the error statistics for the model with and without time-varying volatility. Time variation in volatility does not have a significant effect on the means and standard deviations of the approximation errors. In fact, the error \(e_{\pi,t}\) measured relative to the output gap \(X_t\) has a slightly smaller error under time-varying volatility than under constant volatility. The error \(e_{g,t}\) is larger than the error \(e_{h,t}\) under both specifications, but small. In summary, solving this model with time-varying volatility, log-linearizing around the respective unconditional means, is highly accurate.

4 Conclusion

Time variation in macroeconomic volatility is potentially an important channel to understand some empirical regularities in macroeconomic and asset pricing dynamics. This time variation can be captured by autoregressive gamma processes. This paper incorporates this process to a general specification of dynamic equilibrium models and provides a general solution. Two applications are analyzed in a long-run risks asset pricing model and a simple New Keynesian model. The analysis shows that the solutions are highly accurate, computationally inexpensive, and then very useful for estimation purposes. Given its generality and tractability, the autoregressive gamma process represents an effective tool to incorporate nonnegative variables to equilibrium models.
Figure 2: Comparative statics of loadings for the output gap \((x_t)\) and inflation \((\pi_t)\) on monetary policy shocks \((u_t)\) and the volatility of policy shocks \((v_t)\), by changing \(I_v, \gamma, \omega, \nu, \rho_v, \text{ and } \delta_v\). The loadings are denoted by \(k_j\) where \(k = \{x, \pi\}\) is the variable and \(j = \{u, v\}\) is the shock. The baseline parameter values are presented in Table 4.

### A Proof

The proof relies on the method of undetermined coefficients, where the functional form (4) for the solution is guessed and all coefficients are found to satisfy all expectational equations. Given the dynamics of \(s_t\) in equation (2), the guessed solution for \(z_t\), and the conditional independence between variables \(s_t\) and \(v_t\), equation (3) can be written as

\[
\begin{align*}
\hat{b}_j + b_{j,s}^\top s_t + b_{j,u}^\top u_t + b_{j,v}^\top v_t & = \eta_j \log \mathbb{E}_t \left[ \exp \left( (Z_s^\top d_{j,z} + d_{j,s})^\top s_{t+1} \right) \right] \\
& + \eta_j \log \mathbb{E}_t \left[ \exp \left( (Z_v^\top d_{j,z} + d_{j,v})^\top v_{t+1} \right) \right].
\end{align*}
\]

Given that \(s_t\) is conditionally normally distributed, the second term on the right-hand side of the equation is

\[
\eta_j \log \mathbb{E}_t \left[ \exp \left( (Z_s^\top d_{j,z} + d_{j,s})^\top s_{t+1} \right) \right] = \eta_j (Z_s^\top d_{j,z} + d_{j,s})^\top [\theta_s + \Phi s_t + \Phi s,v v_t] \\
+ \frac{1}{2} \eta_j (Z_s^\top d_{j,z} + d_{j,s})^\top \Sigma(v_t) \Phi_s,\sigma \Sigma(v_t)^\top (Z_s^\top d_{j,z} + d_{j,s}),
\]

(23)
where the diagonal conditional covariance matrix \( \Sigma(\mathbf{v}_t) = \Sigma^{1/2}(\mathbf{v}_t) \Sigma^{1/2}(\mathbf{v}_t)^\top \) can be written as \( \Sigma(\mathbf{v}_t) = \Sigma + \text{diag}(\{\Sigma_v\}) \). That is, the variance and covariance components are linear combinations of the state variables \( \mathbf{v}_t \).

The \( N_x \times N_x \) matrix \( \Sigma \) contains the constant volatility components, and the \( N_x \times N_x \) matrix \( \Sigma_v \) contains the sensitivity of the conditional volatilities to the state variables \( \mathbf{v}_t \).

It follows that

\[
(Z^\top_v d_{j,z} + d_{j,s})^\top \Phi_{s,o}(Z^\top_v d_{j,z} + d_{j,s}) = (Z^\top_v d_{j,z} + d_{j,s})^\top \Phi_{s,o} \Sigma_{s,o} (Z^\top_v d_{j,z} + d_{j,s}) + \psi \left( \Phi_{s,o} (Z^\top_v d_{j,z} + d_{j,s}) \right)^\top \Sigma_v \mathbf{v}_t, \tag{25}
\]

where \( \psi(x) = \text{diag}(\{\text{diag}(x)^2\})^\top \).

Given that \( \mathbf{v}_t \) follows the multivariate autoregressive gamma process in equation (1), the third term on the right-hand side of equation (23) is

\[
\eta_j \log E_t \left[ \exp \left( (Z^\top_v d_{j,z} + d_{j,v})^\top \mathbf{v}_{t+1} \right) \right] = \eta_j g(Z^\top_v d_{j,z} + d_{j,v}) + h(Z^\top_v d_{j,z} + \eta_j d_{j,v})^\top \mathbf{v}_t, \tag{26}
\]

where

\[
g(u) = -\sum_{i=1}^{N_v} \delta_i \log(1 - u_i),
\]

and the \( i \)-th component of the \( N_v \)-vector \( h(u) \) is \( \frac{u_i \delta_i}{1 - \sum \delta_i} \). The scalar \( u_i \) is the \( i \)-th component of vector \( u \).

From equations (24), (25), and (26), equation (23) can be written as

\[
\begin{align*}
\dot{\mathbf{b}}_j &+ (b_{j,z} - \eta_j Z^\top_v d_{j,z})^\top (\mathbf{z} + Z \mathbf{z}_{t-1} + Z \mathbf{v}_t + Z \mathbf{v}_t) + B_2 \mathbf{z}_{t-1} + B_2 \mathbf{v}_t + b_{j,z} \mathbf{v}_t \\
&= \eta_j \left\{ d_{j,z} \mathbf{e} + (Z^\top_v d_{j,z} + d_{j,v})^\top [b_{i,s} + \Phi_{s,s} + \Phi_{s,v} \mathbf{v}_t] + \frac{1}{2} (Z^\top_v d_{j,z} + d_{j,v})^\top \Phi_{s,o} \Sigma_{s,o} (Z^\top_v d_{j,z} + d_{j,s}) \\
&+ \frac{1}{2} \psi \left( \Phi_{s,o} (Z^\top_v d_{j,z} + d_{j,s}) \right)^\top \Sigma_v \mathbf{v}_t + g(Z^\top_v d_{j,z} + d_{j,v}) + h(Z^\top_v d_{j,z} + d_{j,v})^\top \mathbf{v}_t \right\}. \tag{27}
\end{align*}
\]

The coefficients \( \mathbf{z}, Z_z, Z_s, \) and \( \mathbf{Z}_v \) have to satisfy the \( N_z \) equations of the form (27). These coefficients are found using the method of undetermined coefficients. Consider first the coefficients loading on the lagged endogenous variables \( z_{t-1} \). These coefficients have to satisfy

\[
(b_{j,z} - \eta_j Z^\top_v d_{j,z})^\top \mathbf{z}_s + b_{j,z} \mathbf{v}_t = 0.
\]

Denote by \( \mathbf{X} = \{\mathbf{x}_j\}_j \) the matrix whose \( j \)-th row is the vector \( \mathbf{x}_j^\top \) for \( j = \{1, 2, ..., n\} \). Using the notation for the \( N_x \times N_x \) matrices \( B_z = \{b_{j,z}\}_j, D_z = \{\eta_j d_{j,z}\}_j, \) and \( B_{st} = \{b_{j,z}\}_j \), the \( N_z \) equations of the above form can be written as the quadratic matrix equation

\[
B_z \mathbf{Z}_s = D_z \mathbf{Z}_s^2 + B_{st} = 0.
\]

Solving this system provides the \( N_x \times N_x \) coefficients \( \mathbf{Z}_s \). If the number of predetermined endogenous variables is the same as the total number of endogenous variables \( N_z \), the solution can be found using the methods described in McCallum (1983) or Uhlig (1995). If the number of predetermined variables is less than \( N_z \), the solution of the quadratic matrix equation can be found numerically, under the restriction that if all the elements in a column of \( B_{st} \) are zero, all the elements in the corresponding column in \( \mathbf{Z}_s \) are zero too. Using the minimum state variable criterion, \( \mathbf{Z}_s = \mathbf{0} \) if \( B_{st} \equiv \mathbf{0} \).

The coefficients on \( \mathbf{s}_t \) in equation (27) satisfy

\[
(b_{j,z} - \eta_j Z^\top_v d_{j,z})^\top \mathbf{z}_s + b_{j,z} = \eta_j (d_{j,z} \mathbf{Z}_s + d_{j,v} \Phi_{s,v}) \Phi_{s,s},
\]

for all \( j \). Define the \( N_x \times N_x \) matrices \( B_s = \{b_{j,s}\}_j \), and \( D_s = \{\eta_j d_{j,s}\}_j \). The coefficients \( \mathbf{Z}_s \) are found by solving the system of \( N_x \times N_x \) linear equations implied by

\[
(B_s - D_s \mathbf{Z}_s) \mathbf{Z}_s = D_s \mathbf{Z}_s \Phi_{s,v} = D_s \Phi_{s,v} - B_s.
\]

\footnote{The covariance matrix does not need to be diagonal. We assume a diagonal matrix to facilitate the exposition of the results.}
The solution for the $N_z \times N_e$ coefficients in $Z_v$ is given by

$$\text{vec}(Z_v) = \left(I_{N_e} \otimes (B_z - D_z Z_v) - \Phi_s^\top \otimes D_z\right)^{-1} \text{vec}(D_z \Phi_s - B_z). \quad (28)$$

Similarly, coefficients multiplying the state variables $v_t$ in equation (27) imply

$$(b_{j,z} - \eta_j Z_v^\top d_{j,z})^\top Z_v + b_{j,v}^\top v = \eta_j \left[(Z_v^\top d_{j,z} + d_{j,s})^\top \Phi_s v + \frac{1}{2} \psi(Z_v^\top d_{j,z} + d_{j,s})^\top \Sigma v + h(Z_v^\top d_{j,z} + d_{j,s})^\top \right].$$

Define the $N_x \times N_v$ matrices $B_v = \{b_{j,v}\}_{j=1}^{N_e}$, $D_v = \{d_{j,v}\}_{j=1}^{N_e}$, $\Psi_v(\eta, Z_v, D_z, D_v) = \{\eta_j \psi(\Phi_s^\top (Z_v^\top d_{j,z} + d_{j,s})^\top)\}_{j=1}^{N_e}$, and $H(\eta, Z_v, D_z, D_v) = \{\eta_j h(Z_v^\top d_{j,z} + d_{j,s})^\top\}_{j=1}^{N_e}$, where $\eta = (\eta_1, \eta_2, \ldots, \eta_{N_e})^\top$. Using this notation, the coefficients $Z_v$ are found by solving the system of $N_x \times N_v$ quadratic equations implied by

$$(B_z - D_z Z_v)Z_v + B_v = (D_z Z_v + D_v)\Phi_s v + \frac{1}{2} \Psi_v(\eta, Z_v, D_z, D_v)\Sigma_v + H(\eta, Z_v, D_z, D_v).$$

Using the minimum state variable criterion, $Z_v = 0$ if $\Phi_s v = \Sigma_v = B_v = D_v \equiv 0$.

Finally, the constant coefficients in equation (27) imply

$$\dot{b}_j + (b_{j,z} - \eta_j Z_v^\top d_{j,z})^\top \bar{z} = \eta_j \left[d_{j,z}^\top \bar{z} + (Z_v^\top d_{j,z} + d_{j,s})^\top \theta_s + \frac{1}{2} (Z_v^\top d_{j,z} + d_{j,s})^\top \Phi_s \Sigma \Phi_s^\top (Z_v^\top d_{j,z} + d_{j,s}) + g(Z_v^\top d_{j,z} + d_{j,s})\right].$$

Define the $N_x$-vectors $\bar{B} = \{b_{j}\}_{j=1}^{N_e}$, $G(\eta, Z_v, D_z, D_v) = \{\eta_j g(Z_v^\top d_{j,z} + d_{j,s})^\top\}_{j=1}^{N_e}$, and $\bar{\Psi}(\eta, Z_v, D_z, D_v) = \{\eta_j (Z_v^\top d_{j,z} + d_{j,s})^\top \Phi_s \Sigma \Phi_s^\top (Z_v^\top d_{j,z} + d_{j,s})\}_{j=1}^{N_e}$. The coefficients $\bar{z}$ are found by solving the system of $N_x$ linear equations implied by

$$(B_z - D_z Z_v - D_v)\bar{z} + \bar{B} = (D_z Z_v + D_v)\theta_s + \frac{1}{2} \bar{\Psi}(\eta, Z_v, D_z, D_v) + G(\eta, Z_v, D_z, D_v).$$

### B Linearization Points and Iteration Procedure

Consider $N_m$ linearization points $\bar{m}_k$, for $k = \{1, 2, \ldots, N_m\}$, described by the unconditional expectation

$$\bar{m}_k = \mathbb{E} \left[ \bar{a}_k + a_{k,z}^\top z_t + a_{k,s}^\top s_{t-1} + a_{k,v}^\top v_t + \kappa_s \log \mathbb{E} \left[ \exp(c_{k,z}^\top z_{t+1} + c_{k,s}^\top s_{t+1} + c_{k,v}^\top v_{t+1}) \right] \right]. \quad (29)$$

where all vectors multiplying the model variables have appropriate dimensions. Given the model solution, the linearization point becomes

$$\bar{m}_k = \mathbb{E} \left[ \bar{a}_k + (a_{k,z} + \kappa Z_v^\top c_{k,z}^v) z_t + a_{k,s}^\top s_{t-1} + a_{k,v}^\top v_t + \kappa_c c_{k,v}^v \bar{z} + \kappa(k(Z_v^\top c_{k,z} + c_{k,s})^\top \mathbb{E}[s_{t+1}] + \frac{1}{2} \kappa(k(Z_v^\top c_{k,z} + c_{k,s})^\top \Phi_s \Sigma (\Sigma + \text{diag}(\Sigma_v)) \Phi_s^\top (Z_v^\top c_{k,z} + c_{k,s})) + \kappa g(Z_v^\top c_{k,z} + c_{k,v}) + \kappa h(Z_v^\top c_{k,z} + c_{k,v})^\top v_t \right]$$

$$= \bar{a}_k + (a_{k,z} + \kappa Z_v^\top c_{k,z} + a_{k,s})^\top \mathbb{E}[z_t] + \kappa_c c_{k,v}^v \bar{z} + \mathbb{E}[s_{t+1}] + \frac{1}{2} \kappa(k(Z_v^\top c_{k,z} + c_{k,s})^\top \Phi_s \Sigma \Phi_s^\top (Z_v^\top c_{k,z} + c_{k,s}) + \kappa g(Z_v^\top c_{k,z} + c_{k,v}) + \kappa h(Z_v^\top c_{k,z} + c_{k,v})^\top v_t \right]$$

$$+ \left(a_{k,v} + \frac{1}{2} \kappa \psi(\Phi_s^\top (Z_v^\top c_{k,z} + c_{k,s}))) \Sigma v + \kappa_s h(Z_v^\top c_{k,z} + c_{k,v}) \right)^\top \mathbb{E}[v_{t+1}] + \mathbb{E}[v_t],$$

where $\mathbb{E}[s_{t+1}] = (I_{N_s} - \Phi_s)^{-1} \theta_s, \mathbb{E}[v_{t+1}] = (I_{N_v} - \Phi_v)^{-1} (\bar{z} + Z_v \mathbb{E}[s_{t+1}] + Z_v \mathbb{E}[v_t]) + \mathbb{E}[v_t]$. In matrix form, $\bar{M} = (\bar{m}_1, \bar{m}_2, \ldots, \bar{m}_{N_m})^\top$ is given by
\[ \tilde{M} = \tilde{A} + (A_2 + C_{21}Z_2 + A_{31})E[z_1] + C_2Z_2 + \frac{1}{2} \tilde{\Psi}(\kappa, Z_2, C_2, C_3) + G(\kappa, Z_2, C_2, C_3) + \frac{1}{2} \Psi_\sigma(\kappa, Z_2, C_2, C_3) + H(\kappa, Z_2, C_2, C_3) \] + \left( A_2 + C_{21}Z_2 + C_{31} \right) E[s_1] + \left( A_v + \frac{1}{2} \Psi_\sigma(\kappa, Z_2, C_2, C_3) + H(\kappa, Z_2, C_2, C_3) \right) \operatorname{E}[v_t],

where

\[ A = \left\{ a_k \right\}_{k=1}^{N_m} \right]_{N_m \times N_z}, \quad A_2 = \left\{ a_k \right\}_{k=1}^{N_m} \right]_{N_m \times N_z}, \quad A_3 = \left\{ a_k \right\}_{k=1}^{N_m} \right]_{N_m \times N_z}, \quad A_v = \left\{ a_k \right\}_{k=1}^{N_m} \right]_{N_m \times N_v}, \]

\[ \Psi(\kappa, Z_2, C_2, C_3) = \left\{ \kappa_k \right\}_{k=1}^{N_m} \left( Z_2^{T} c_k + c_k \right) \right]_{N_m \times N_z}, \quad \Psi_\sigma(\kappa, Z_2, C_2, C_3) = \left\{ \kappa_k \right\}_{k=1}^{N_m} \left( Z_2^{T} c_k + c_k \right) \right]_{N_m \times N_v}, \]

\[ G(\kappa, Z_2, C_2, C_3) = \left\{ \kappa_k \right\}_{k=1}^{N_m} \left( Z_2^{T} c_k + c_k \right) \right]_{N_m \times N_z}, \quad H(\kappa, Z_2, C_2, C_3) = \left\{ \kappa_k \right\}_{k=1}^{N_m} \left( Z_2^{T} c_k + c_k \right) \right]_{N_m \times N_v}, \]

for \( \kappa = (\kappa_1, \kappa_2, ..., \kappa_{N_m})^T \). For an initial guess for all \( \bar{m}_k \)'s affecting equations (23), the system of equations is solved, and a new set \( M \) is found. The procedure is repeated until convergence. That is, until \( M^{(n+1)} \approx M^{(n)} \). Given the particular nature and parameter values of the model, it is possible that a fixed point cannot be found. In this case, the iteration procedure can be stopped at iteration \( n \) if \( \| M^{(n+1)} - M^{(n)} \| > \| M^{(n)} - M^{(n-1)} \| \). That is, if the divergence between previous and new linearization points increases.

### C Long-Run Risks Model

#### C.1 Solution

The system of expectational equations is described by (9) for \( q = \{c, d\} \), and (12). Notice that these equations conform to the functional form (3). The system can be described in terms of the solution method in Section 2 by the state variables \( y_t = \left( \Delta c_t, x_t, \Delta d_t \right)^T \) and \( v_t = v_t \), and the endogenous variables \( z_t = \left( p_{c,t}, p_{d,t}, r_t \right)^T \). The vectors and matrices to find the solution become \( \theta_t = \left( \mu_c, 0, \mu_d \right)^T \),

\[ \Phi_x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \phi_x & 0 \\ 0 & \phi_{dx} & 0 \end{bmatrix}, \quad \Phi_{x,\sigma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sigma_{dc} & 0 & 1 \end{bmatrix}, \]

\footnote{In fact, the risk-free rate and its corresponding equation (12) do not need to be included in the system to find the solution. However, including them allows us to compute easily the sensitivity of the risk-free rate to the state variables and shocks in the economy. A similar exercise can be done including the stock return as an endogenous variable and adding equation (11) to the system.}
\( \Phi_{s,v} = \Phi_{3 \times 1}, \Sigma = (1 - I_v) \text{diag}(\sigma_e^2, \sigma_v^2, \sigma_d^2) \), and \( \Sigma_v = I_v(\sigma_e^2, \sigma_v^2, \sigma_d^2)\). Let 

\[
\begin{bmatrix}
-\theta \log \beta - \theta \tilde{\eta}_c \\
-\theta \log \beta + (1 - \theta) \tilde{\eta}_c - \tilde{\eta}_d \\
-\theta \log \beta + (1 - \theta) \tilde{\eta}_c
\end{bmatrix}, \quad
\begin{bmatrix}
\theta & 0 & 0 \\
-(1 - \theta) & 1 & 0 \\
-(1 - \theta) & 0 & -1
\end{bmatrix},
\]

\( B_x = B_x = 0_{3 \times 3}, \quad B_v = 0_{3 \times 1}, \)

\( D_x = \begin{bmatrix}
\theta \eta_c & 0 & 0 \\
-(1 - \theta) \eta_c & \eta_d & 0 \\
-(1 - \theta) \eta_c & 0 & 0
\end{bmatrix}, \quad \text{and} \quad D_x = \begin{bmatrix}
1 - \gamma & 0 & 0 \\
-\gamma & 0 & 1 \\
-\gamma & 0 & 0
\end{bmatrix}. \)

\( \Phi_{s,v} = 0_{3 \times 1}. \)

Equation (10) for the endogenous linearization points has the functional form in Appendix B. The matrices for the linearization points \( M = (\tilde{m}_e, \tilde{m}_d) \) are

\[
\begin{bmatrix}
\theta \log \beta + \theta \tilde{\eta}_c \\
\theta \log \beta - (1 - \theta) \tilde{\eta}_c + \tilde{\eta}_d \\
\theta \log \beta - (1 - \theta) \tilde{\eta}_c
\end{bmatrix}, \quad
\begin{bmatrix}
1 - \theta & 0 & 0 \\
1 - \theta & 0 & 0 \\
1 - \theta & 0 & 0
\end{bmatrix}, \quad A_{zd} = A_x = 0_{2 \times 3},
\]

\[
A_v = 0_{2 \times 1}, \quad C_z = \begin{bmatrix}
\theta \eta_c & 0 & 0 \\
-(1 - \theta) \eta_c & \eta_d & 0
\end{bmatrix}, \quad \text{and} \quad C_x = \begin{bmatrix}
1 - \gamma & 0 & 0 \\
-\gamma & 0 & 1
\end{bmatrix}.
\]

### C.2 Risk Premia Under Normal and Autoregressive Gamma Processes

Let \( m_{s,t+1} = m_{s,t+1} + m_v,t+1 \) be the log-pricing kernel, where

\[
m_{s,t+1} = E_t[m_{s,t+1}] - \lambda^T \Sigma(v_i)^{1/2} \varepsilon_{t+1}, \quad \text{and} \quad m_v,t+1 = E_t[m_v,t+1] - \lambda^T (v_{t+1} - E_t[v_{t+1}]),
\]

are the conditional normal and autoregressive gamma process components, respectively. The prices of Gaussian risk are contained in the \( \lambda_e \) vector and the prices of autoregressive gamma risk are contained in the \( \lambda_v \) vector. Similarly, let \( r_{s,t+1} = r_{s,t+1} + r_v,t+1 \) be the continuously compounded asset return where

\[
r_{s,t+1} = E_t[r_{s,t+1}] + r_{s,t+1}^T \Sigma(v_i)^{1/2} \varepsilon_{t+1}, \quad \text{and} \quad r_v,t+1 = E_t[r_v,t+1] + r_{v,t+1}^T (v_{t+1} - E_t[v_{t+1}]),
\]

are the conditional normal and autoregressive components, respectively, where \( r_s \) and \( r_v \) are the vectors of return sensitivities to Gaussian and autoregressive gamma risks, respectively.

From the pricing equation \( 1 = E_t[\exp(m_{s,t+1} + r_{s,t+1})] \), we can characterize the risk-free rate \( R_{f,t} = \exp(r_{f,t}) \) by

\[
\frac{1}{R_{f,t}} = E_t[M_{t+1}] = \exp \left( E_t[m_{t+1}] + \frac{1}{2} \text{var}(m_{s,t+1}) \right) \times \exp \left( \sum_{i=1}^{N_v} \delta_i [\lambda_{v,i} \xi_i - \log(1 + \lambda_{v,i} \xi_i)] + \sum_{i=1}^{N_v} \lambda_{v,i} \rho_i \left( 1 - \frac{1}{1 + \lambda_{v,i} \xi_i} \right) v_{i,t} \right).
\] (30)

The asset return \( R_t = \exp(r_t) \) has expected return

\[
E_t[R_{t+1}] = \exp \left( E_t[r_{t+1}] + \frac{1}{2} \text{var}(r_{s,t+1}) \right) \times \exp \left( - \sum_{i=1}^{N_v} \delta_i [r_{v,i} \xi_i + \log(1 + r_{v,i} \xi_i)] - \sum_{i=1}^{N_v} r_{v,i} \rho_i \left( 1 - \frac{1}{1 - r_{v,i} \xi_i} \right) v_{i,t} \right).
\] (31)
The asset return also satisfies the equation

\[
\mathbb{E}_t[M_{t,t+1}R_{t+1}] = \exp \left( \mathbb{E}_t[m_{t+1} + r_{t+1}] + \frac{1}{2} \text{var}_t(m_{s,t+1} + r_{s,t+1}) \right)
\]

\times \exp \left( \sum_{i=1}^{N_v} \delta_i \left[ (\lambda_{v,i} - r_{v,i})\varsigma_i - \log(1 + (\lambda_{v,i} - r_{v,i})\varsigma_i) \right] \right)

\times \exp \left( - \sum_{i=1}^{N_v} (\lambda_{v,i} - r_{v,i})\rho_i \left( 1 - \frac{1}{1 + (\lambda_{v,i} - r_{v,i})\varsigma_i} \right)^{v_i,t} \right).

Reorganizing terms and replacing equations (30) and (31) in equation (32), the asset return premium is

\[
\log \left( \frac{\mathbb{E}_t[R_{t+1}]}{R_{f,t}} \right) = -\text{cov}_t(m_{s,t+1}, r_{s,t+1}) + \sum_{i=1}^{N_v} \delta_i \log \left[ \frac{1 + (\lambda_{v,i} - r_{v,i})\varsigma_i}{(1 + \lambda_{v,i}\varsigma_i)(1 - r_{v,i}\varsigma_i)} \right]

+ \sum_{i=1}^{N_v} \rho_i \left( \frac{\lambda_{v,i} - r_{v,i}}{1 + (\lambda_{v,i} - r_{v,i})\varsigma_i} - \frac{\lambda_{v,i}}{1 + \lambda_{v,i}\varsigma_i} + \frac{r_{v,i}}{1 - r_{v,i}\varsigma_i} \right) v_i,t.
\]

Notice that the risk premium also can be written in terms of the continuously compounded excess returns as

\[
\log \left( \frac{\mathbb{E}_t[R_{t+1}]}{R_{f,t}} \right) = \mathbb{E}_t[r_{t+1}] - r_{f,t} + J.I.t,
\]

where, from equation (31), the Jensen’s inequality term is

\[
J.I.t = \frac{1}{2} \text{var}_t(r_{s,t+1}) - \sum_{i=1}^{N_v} \delta_i \left[ r_{v,i}\varsigma_i + \log(1 - r_{v,i}\varsigma_i) \right] - \sum_{i=1}^{N_v} \frac{r_{v,i}^2\rho_i\varsigma_i}{1 - r_{v,i}\varsigma_i} v_i,t.
\]

D New Keynesian Model

D.1 Firm’s Problem

Under monopolistic competition and Calvo (1983) staggered price setting, a firm can choose the optimal \(P_t^*\) with probability \(\alpha\) each period. The firm’s optimization problem is

\[
\max_{P_t^*} \mathbb{E}_t \left[ \sum_{s=0}^{\infty} \alpha^s M_{t,t+s} (P_t^* Y_{t+s}(j) - W_{t+s}(j)) N_{t+s}(j) \right]
\]

s.t. \(Y_{t+s}(j) = A_{t+s} N_{t+s}(j)\)

\(Y_{t+s}(j) = \left( \frac{P_t^*}{P_{t+s}} \right)^{-\theta} Y_{t+s}\)

\(P_t = \left[ \int_0^1 P_t(j)^{1-\theta} dj \right]^{\frac{1}{1-\theta}} = \left(1 - \alpha\right)(P_t^*)^{1-\theta} + \alpha P_{t-1}^{1-\theta} \right]^{\frac{1}{1-\theta}} \).

where \(M\) is the pricing kernel, \(Y\) is output, \(W\) is the nominal wage, and \(N\) is labor demand. The subscript \(t+s|t\) denotes the value in period \(t+s\) given that the last price adjustment was in period \(t\). The problem constraints are the production function, where \(A\) is labor productivity, and the product demand curve. The first order condition for the firm is:

\[
\mathbb{E}_t \left[ \sum_{s=0}^{\infty} \alpha^s M_{t,t+s} (P_t^* Y_{t+s}(j) - \mu S_{t+s}(j)) \right] = 0,
\]

18
where \( \mu = \frac{\theta}{\pi} \) is the frictionless markup in the absence of price rigidities, and \( S_{t+1|t} \) is the marginal (nominal) cost

\[
S_{t+1|t} = \frac{Y_{t+1|t}^{1+\omega}}{A_{t+1|t}^{1+\omega}} Y_{t+1} \frac{P_{t}}{P_{t+1}}.
\]

(39)

From (36) and (39), (38) can be written as

\[
E_t \left[ \sum_{s=0}^{\infty} \alpha^s M_{t,t+s} Y_{t+s} \left( \frac{P_t^*}{P_{t+s}} \right)^{-\theta} P_{t+s} \right] = E_t \left[ \sum_{s=0}^{\infty} \alpha^s M_{t,t+s} \mu \left( \frac{Y_{t+1|t}^{1+\omega}}{A_{t+1|t}^{1+\omega}} \right) \left( \frac{P_t}{P_{t+s}} \right)^{-\theta} P_{t+s} \right].
\]

The left-hand side of the equation can be written as

\[
P_t^* \left( \frac{P_t}{P_t} \right)^{-\eta} Y_t H_t, \text{ where } H_t = E_t \left[ \sum_{s=0}^{\infty} \alpha^s M_{t,t+s} \left( \frac{Y_{t+s}}{Y_t} \right) \left( \frac{P_t}{P_{t+s}} \right)^{-\theta} \right],
\]

and the right-hand side of the equation can be written as

\[
\mu Y_t^{1+\omega+\gamma} \left( \frac{P_t^*}{P_t} \right)^{-\theta} \left( \frac{P_t}{P_{t+s}} \right) G_t,
\]

where

\[
G_t = E_t \left[ \sum_{s=0}^{\infty} \alpha^s M_{t,t+s} \left( \frac{A_t}{A_{t+s}} \right)^{1+\omega} \left( \frac{Y_{t+s}}{Y_t} \right)^{1+\omega+\gamma} \left( \frac{P_t}{P_{t+s}} \right)^{-\theta-1} \right].
\]

(40)

(41)

\( H_t \) and \( G_t \) can be written recursively as in equations (17) and (18), respectively. Output is \( Y_t = Y_t^f X_t \), where the natural rate of output satisfies \( \mu(Y_t^f)^{\omega+\gamma} = A_t^{1+\omega} \). It follows that the optimality condition can be written as

\[
\left( \frac{P_t^*}{P_t} \right) H_t = X_t^{\omega+\gamma} G_t.
\]

Finally, equation (37) can be written

\[
\frac{P_t^*}{P_t} = \left[ \frac{1}{1-\alpha} \left( 1 - \alpha \Pi_t^{-(1-\theta)} \right) \right]^{1/\gamma},
\]

to obtain (15).

### D.2 Solution

The solution of this system implies linear solutions for \( x_t, \pi_t, h_t, \) and \( g_t \) that depend on policy shocks and the volatility of the shocks. Notice that equations (19) and (20) have the form of the expectational equation (3). The solution of the system can then be obtained using the method in Section 2. In particular, let \( z_t = (x_t, \pi_t, h_t, g_t) \), \( s_t = u_t \), and \( v_t = \nu_t \). The system of expectational equations is given by (21), (19), (20), and (14), where the interest rate \( i_t \) is replaced with the right-side of equation (16).

The vector and matrices in Section 2 become

\[
\hat{B} = \begin{bmatrix} \tilde{\eta}_\pi \\ \tilde{\eta}_h \\ \tilde{\eta}_\theta \\ -\tilde{L} \log \beta \end{bmatrix}, \quad \hat{B}_z = \begin{bmatrix} -(\omega + \gamma) & \eta_\pi & 1 & -1 \\ \eta_\pi(1+\gamma) & 0 & 1 & 0 \\ \eta_\theta(1+\omega) & 0 & 0 & 1 \\ -\gamma & -\eta_\pi & 0 & 0 \end{bmatrix}, \quad \hat{B}_z = \mathbf{0}_{4 \times 1}, \quad \hat{B}_s = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{B}_v = \mathbf{0}_{4 \times 1},
\]

\[
\hat{D}_z = \begin{bmatrix} 0 & \eta_\pi(1+\gamma) & \eta_\pi(\theta-1) & 0 \\ \eta_\pi(1+\gamma) & 0 & 0 & \eta_\theta \\ -\gamma & -\eta_\pi & 0 & 0 \end{bmatrix}, \quad \hat{D}_s = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{D}_v = \mathbf{0}_{4 \times 1},
\]

19
\[ \Sigma = (1 - I_v)\sigma_a^2, \text{ and } \Sigma_v = I_v\sigma_a^2. \]

The linearization points are the unconditional expectations
\[
\begin{align*}
\bar{m}_\pi &= \mathbb{E}[\pi_t], \\
\bar{m}_h &= \mathbb{E}\left[\log \mathbb{E}[e^{(1-\gamma)\Delta x_{t+1} + (\theta-1)\pi_{t+1} + h_{t+1}}]\right], \\
\bar{m}_g &= \mathbb{E}\left[\log \mathbb{E}[e^{(1+\omega)\Delta x_{t+1} + \theta\pi_{t+1} + g_{t+1}}]\right].
\end{align*}
\]

These equations have the form of equation (29). The vectors and matrices associated to the linearization points become
\[
\begin{align*}
\bar{A} &= 0_{3 \times 1}, \\
A_z &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(1 - \gamma) & 0 & 0 & 0 \\ -(1 + \omega) & 0 & 0 & 0 \end{bmatrix}, \quad A_zl = A_s = A_v = 0_{3 \times 1}, \\
C_z &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ (1 - \gamma) & (\theta - 1) & 1 & 0 \\ (1 + \omega) & \theta(1 + \omega) & 0 & 1 \end{bmatrix}, \quad \text{and} \quad C_z = C_v = 0_{3 \times 1}.
\end{align*}
\]
References


