A Simple Solution Method for Dynamic Equilibrium Models with Time-Varying Volatility[*]

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Abstract

Dynamic economic models should capture well-documented evidence on time-varying volatility of several economic and financial series. We present a computationally-inexpensive general solution method for dynamic equilibrium models where volatility follows an autoregressive gamma process. The process implies linear equilibrium dynamics and provides moments in closed-form. As applications, we compute the volatility premium in the Bansal and Yaron (2004) asset pricing model, and impulse responses to volatility shocks in a New Keynesian model. The accuracy of the two model solutions is not negatively affected by time-varying volatility.

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1 Introduction

Time variation in the volatility of several macroeconomic and financial time series has been widely documented since the seminal contribution of Engle (1982).[1] This has motivated recent efforts to capture time-varying volatility in equilibrium models and understand its implications on macroeconomic and asset pricing dynamics. For instance, Justiniano and Primiceri (2008) find a significant role for time-varying volatility in explaining the "Great Moderation," and Bansal and Yaron (2004) suggest that financial asset returns incorporate a premium for variation in economic uncertainty. However, adding time-varying volatility to equilibrium models is not an easy task. Model solutions often rely on numerical methods and/or high-order approximations than can be difficult to analyze and can be challenging for estimation purposes.[2] Alternatively, some studies obtain closed-form solutions under the inadequate assumption of a volatility process that can take negative values. We study a general specification for discrete-time equilibrium models where time-varying volatility is incorporated as the tractable non-negative autoregressive gamma process. We provide a simple solution method for these models and present two applications of the method and its accuracy.

Discrete-time dynamic stochastic models are characterized by sets of expectational equations of the form

$$\mathbb{E}_t \left[f \left(\mathbf{u}_{t+1}, \mathbf{u}_t \right) \right] = 0. \tag{1}$$

Recursive Bellman equations, equilibrium Euler equations, and no-arbitrage equations, for

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[1] See Stock and Watson (2003) and Sims and Zha (2006) for evidence on macroeconomic dynamics. See Bollerslev, Chou and Kroner (1992) for a survey on evidence of time varying volatility in financial series.

[2] See Fernández-Villaverde and Uribe (2011) for a Bayesian estimation of a DSGE model with time-varying volatility using a particle filter.

instance, can be written in this form, where $f(\cdot)$ is a function of a set $\mathbf{u}_t$ of endogenous and exogenous variables. The model solution consists in finding the dynamics of endogenous variables that satisfy the expectational equations given pre-specified processes for the exogenous variables. Usually, the non-linearity of $f(\cdot)$ does not allow us to obtain exact closed-form solutions for the model. We rely then on numerical solutions of the system, or approximate analytical solutions whose accuracy depend on the approximation method. For tractability, exogenous variables are commonly assumed to follow conditionally normal distributions. However, a normal distribution is not a reasonable assumption if our purpose is to model the process of non-negative variables such as volatility. A more appropriate, and still tractable, characterization for non-negative variables is provided by the autoregressive gamma process in Jasiak and Gourieroux (2006). This process is the discrete counterpart of the Cox, Ingersoll and Ross (1985) process. Its moment generating function is available in closed-form and its conditional first and second moments are linear functions of the process. We analyze a general model specification where uncertainty is captured by normal and autoregressive gamma processes.

We present the general framework in Section 2 along with a summary of the solution method and associated proof. The method has several advantages. The solution for the dynamics of the endogenous variables is a linear function of exogenous and predetermined endogenous variables. Finding the coefficients in these functions only involves solving linear and quadratic equations, and requires a small amount of computational resources and time. The linearity of the solution allows us to compute moments in closed-form that can be easily compared to the data counterparts.

The framework is general enough to cover a wide range of models in economics and finance previously analyzed in the literature. Depending on the model, exact or log-linearized model equations can be used to satisfy the particular functional form required by the method. If the equations are log-linearized, the accuracy of the solution depends on the linearization points used for the approximation. We show how to obtain endogenous linearization points as unconditional means of appropriate variables by solving a fixed point problem. This choice of linearization points is reasonable to reduce approximation errors that can be characterized and evaluated statistically.

Section 3 presents two applications of the method. The first application is a modified version of the Bansal and Yaron (2004) asset pricing model. The main difference with respect to their framework is related to the choice to model volatility. While they model volatility as an approximate discrete square root process, we model it as an autoregressive gamma process. The approximate square root process has the disadvantage that it allows for negative values of volatility. This can be a minor or significant problem depending on the parameter values for volatility. For instance, Beeler and Campbell (2009) use simulations to show that the Bansal and Yaron calibration involves negative volatility values only 0.001% of the time. However, the specification in Bansal, Kiku and Yaron (2009) generates negative values for volatility more that 5% of the time. The autoregressive gamma process avoids this inconvenience since it is only defined over positive values. We calibrate the autoregressive gamma process parameters to make them comparable to the volatility parameters in the Bansal and Yaron specification. In most dimensions, the two models generate very similar results for the dynamics of macroeconomic and financial variables. However, the volatility premium implied by the autoregressive gamma process is smaller than the premium implied by the approximate square root process. The difference becomes quantitatively important for high persistence and volatility in volatility shocks. We also show that the errors involved in the log-linearization of the model equations are small. A statistical analysis of these errors based on the Den Haan and Marcet (1994) method shows that having time-varying volatility in the model does not particularly affect the quality of the approximation.

The second application is a New Keynesian model similar to a simple version of the one presented in Clarida, Galí and Gertler (2000). Uncertainty in the model is captured by policy shocks with autoregressive gamma volatility. We compare the traditional linearization method with our log-linearization method to find the solution . The traditional method ignores the time variation in volatility, and involves a solution that does not differ from the one for an economy with constant volatility. The log-linearization method shows negative and positive responses of output and inflation, respectively, to a positive shock in volatility. Therefore, the solution method is useful to understand economic dynamics resulting from changes in uncertainty related to monetary policy. We analyze the errors implied by the two solution methods with and without time-varying volatility. While the linearization method involves large and volatile errors, the log-linearization provides a more accurate solution. This accuracy is not significantly affected by the presence of time-varying volatility.

This paper is mostly related to the existing literature on solving rational expectation models using linear difference equations. In the seminal paper by Blanchard and Kahn (1980), the authors outline the necessary conditions to derive the optimal decision rules and laws of motion for endogenous state variables using matrix decomposition. More recently, Uhlig (1995) develops a"toolkit" to solve for the recursive equilibrium laws of motion of the system of log-linearized equations by using the method of undetermined coefficients. Our paper builds on Uhlig's toolkit to incorporate a process for time-varying volatility and characterize a general model solution. In addition, while most of the literature has focused on approximations of equilibrium around a non-stochastic steady state, our solution method provides tools to compute the equilibrium around a stochastic steady state, iteratively determined by unconditional expectations of endogenous variables.

This paper also contributes to the burgeoning literature on modeling stochastic volatility in production economies. Justiniano and Primiceri (2008) and Fernández-Villaverde and

Rubio-Ramírez (2007) examine the "Great Moderation" in the U.S. economy by estimating dynamic general equilibrium models with built-in heteroskedastic shocks. Both studies find that time-varying volatility contributes to the reduction of observed standard deviation of output growth in the data. Furthermore, Bloom (2009) concludes that firm level volatility shocks in productivity leads to lower investment and decrease in output. Finally, Fernández-Villaverde and Uribe (2011) allow stochastic volatility of the real interest rate in a small open-economy dynamic general equilibrium model. They find that volatility shocks can generate recessions endogenously. While these papers rely on high order approximations of the equilibrium conditions to capture time varying volatility, our method only requires the exact equilibrium conditions or log-linerarizations that deliver linear laws of motion for the endogenous dynamics.

2 Model Specification and Solution

Let $\mathbf{z}_t$ be a set of N_z endogenous variables, $\mathbf{s}_t$ be a set of N_s conditionally normal exogenous variables, and $\mathbf{v}_t$ be a set of exogenous variables following autoregressive gamma processes. The vector $\mathbf{z}_t$ can contain predetermined and non-predetermined endogenous variables. The dynamics of $\mathbf{s}_t$ is given by

$$\mathbf{s}_{t+1} = \theta_s + \Phi_s \mathbf{s}_t + \Phi_{s,v} \mathbf{v}_t + \Sigma^{1/2}(\mathbf{v}_t)\varepsilon_{t+1}, \tag{2}$$

where $\mathbf{s}_t$ is an N_s-vector, $\mathbf{v}_t$ is an N_v-vector, θ_s is an N_s-vector, Φ_s is the $N_s \times N_s$ matrix of autoregressive coefficients, $\Phi_{s,v}$ is the $N_s \times N_v$ matrix containing the loadings of $\mathbf{s}_t$ on $\mathbf{v}_t$, $\Sigma^{1/2}(\mathbf{v}_t)$ is the $N_s \times N_s$ matrix capturing the potentially time-varying conditional volatility of the state variables, and the N_s-vector ε_{t+1} denotes the independent Gaussian innovations.

That is, $\varepsilon_{t+1} \sim \text{IID}\mathcal{N}(0, \mathbb{I}_{N_s})$, where $\mathbb{I}_x$ denotes the $x \times x$ identity matrix. The state variables $\mathbf{s}_t$ and $\mathbf{v}_t$ are conditionally independent.

The vector of autoregressive gamma processes $\mathbf{v}_t = (v_{1,t}, v_{2,t}, ..., v_{N_v,t})^\top$, has conditionally independent and unconditionally uncorrelated components

$$\begin{aligned} \frac{v_{i,t+1}}{\varsigma_i} | \, (\mathcal{P}, v_{i,t}) &\sim \text{Gamma}(\delta_i + \mathcal{P}), \\ \text{where} \quad \mathcal{P} | v_{i,t} &\sim \text{Poisson}\left(\frac{\rho_i v_{i,t}}{\varsigma_i}\right). \end{aligned} \tag{3}$$

The conditional mean and variance of these components are, respectively,

$$\mathbb{E}_t[v_{i,t+1}] = \delta_i \varsigma_i + \rho_i v_{i,t}, \quad \text{and} \quad \text{var}_t(v_{i,t+1}) = \delta_i \varsigma_i^2 + 2\varsigma_i \rho_i v_{i,t}.$$

It follows that their unconditional mean and variance are, respectively,

$$\mathbb{E}[v_i] = \frac{\delta_i \varsigma_i}{1 - \rho_i}, \quad \text{and} \quad \text{var}(v_i) = \frac{\delta_i \varsigma_i^2}{(1 - \rho_i)^2},$$

and their conditional moment generating function is

$$\mathbb{E}_t \left[\exp(u v_{i,t+1})\right] = \exp\left[-\delta_i \log(1 - u\varsigma_i) + \frac{u \rho_i}{1 - u\varsigma_i} v_{i,t}\right].$$

Jasiak and Gourieroux (2006) present a general analysis of this process and show that, when the time step tends to zero, it converges to the continuous-time Cox, Ingersoll and Ross (1985) process. The process has been recently used by Le, Singleton and Dai (2010) to study the term structure of interest rates.

Consider a system of N_z expectational equations, where equation j in the system can be written as

$$\bar{b}_j + b_{j,z}^\top \mathbf{z}_t + b_{j,zl}^\top \mathbf{z}_{t-1} + b_{j,s}^\top \mathbf{s}_t + b_{j,v}^\top \mathbf{v}_t = \eta_{j,z} \log \mathbb{E}_t \left[\exp \left(d_{j,z}^\top \mathbf{z}_{t+1} + d_{j,s}^\top \mathbf{s}_{t+1} + d_{j,v}^\top \mathbf{v}_{t+1} \right) \right]. \quad (4)$$

This expectational equation can be the result of exact algebraic manipulation or a log-linearization of an equation of the form (1). All coefficients multiplying the model variables are scalars and vectors of appropriate dimensions.

2.1 Summary of the Solution Procedure

The model solution implies a linear process for the endogenous variables given by

$$\mathbf{z}_t = \bar{z} + Z_z \mathbf{z}_{t-1} + Z_s \mathbf{s}_t + Z_v \mathbf{v}_t, \quad (5)$$

where the N_z-vector $\bar{z}$, the $N_z \times N_z$ matrix Z_z, the $N_z \times N_s$ matrix Z_s, and the $N_z \times N_v$ matrix Z_v are coefficients satisfying the system of N_z expectational equations (4). We summarize here the procedure to find all coefficients. The procedure becomes iterative if at least one of the expectational equations is the result of a log-linearization around endogenous linearization points. All matrices, vectors, and functions multiplying the endogenous coefficients $\bar{z}$, Z_z, Z_s, Z_v are characterized in the proof. The procedure steps are:

1. If the expectational equations are the result of log-linearizations, start iteration n with a set of linerization points $\bar{M}^{(n)}$ to replace them in the expectational equations.
2. Find coefficients Z_z that satisfy the quadratic matrix equation

$$D_z Z_z^2 - B_z Z_z - B_{zl} = 0.$$

3. Find coefficients Z_s that satisfy

$$\mathrm{vec}(Z_s) = \left(\mathbb{I}_{N_z \times N_s} - \Phi_s^\top \otimes (B_z - D_z Z_z)^{-1} D_z\right)^{-1} \mathrm{vec}\left((B_z - D_z Z_z)^{-1}(D_s \Phi_s - B_s)\right).$$

   The symbol $\otimes$ denotes the Kronecker product and "vec(X)" denotes the vectorization of matrix X.

4. Find coefficients Z_v from the system of quadratic equations

$$(B_z - D_z Z_z) Z_v + B_v = (D_z Z_s + D_s)\Phi_{s,v} + \frac{1}{2}\Psi_v(\boldsymbol{\eta_z}, Z_s, D_z, D_s)\Sigma_v + H(\boldsymbol{\eta_z}, Z_v, D_z, D_v).$$

5. Find coefficients $\bar{z}$ from the system of linear equations

$$(B_z - D_z Z_z - D_z)\bar{z} + \bar{B} = (D_z Z_s + D_s)\theta_s + \frac{1}{2}\bar{\Psi}(\boldsymbol{\eta_z}, Z_s, D_z, D_s) + G(\boldsymbol{\eta_z}, Z_v, D_z, D_v).$$

6. If the expectational equations are the result of log-linearizations, find the new set of endogenous linearization points

$$\begin{aligned} \bar{M}^{(n+1)} &= \bar{A} + (A_z + C_z Z_z + A_{zl})\mathbb{E}[\mathbf{z}_t] + C_z \bar{z} + \frac{1}{2}\bar{\Psi}(\boldsymbol{\eta_m}, Z_s, C_z, C_s) \\ &+ G(\boldsymbol{\eta_m}, Z_v, C_z, C_v) + (A_s + C_z Z_s + C_s)\,\mathbb{E}[\mathbf{s}_t] \\ &+ \left(A_v + \frac{1}{2}\Psi_v(\boldsymbol{\eta_m}, Z_s, C_z, C_s) + H(\boldsymbol{\eta_m}, Z_v, C_z, C_v)\right)\mathbb{E}[\mathbf{v}_t], \end{aligned}$$

   where $\mathbb{E}[\mathbf{z}_t] = (\mathbb{I}_{N_z} - Z_z)^{-1}\left(\bar{z} + Z_s \mathbb{E}[\mathbf{s}_t] + Z_v \mathbb{E}[\mathbf{v}_t]\right)$.

7. If the expectational equations are the result of log-linearizations, repeat all the steps above until convergence $\left(M^{(n+1)} \approx M^{(n)}\right)$. If convergence cannot be achieved, stop the procedure at iteration n if $\|M^{(n+1)} - M^{(n)}\| > \|M^{(n)} - M^{(n-1)}\|$.

2.2 Proof

The proof relies on the method of undetermined coefficients, where the functional form (5) for the solution is guessed and all coefficients are found to satisfy all expectational equations. Given the dynamics of $\mathbf{s}_t$ in equation (2), the guessed solution for $\mathbf{z}_t$, and the conditional independence between variables $\mathbf{s}_t$ and $\mathbf{v}_t$, equation (4) can be written as

$$\begin{aligned}
\bar{b}_j + b_{j,z}^\top \mathbf{z}_t + b_{j,zl}^\top \mathbf{z}_{t-1} + b_{j,s}^\top \mathbf{s}_t + b_{j,v}^\top \mathbf{v}_t &= \eta_{j,z} d_{j,z}^\top (\bar{z} + Z_z \mathbf{z}_t) \\
&+ \eta_{j,z} \log \mathbb{E}_t \left[\exp\left((Z_s^\top d_{j,z} + d_{j,s})^\top \mathbf{s}_{t+1}\right)\right] \\
&+ \eta_{j,z} \log \mathbb{E}_t \left[\exp\left((Z_v^\top d_{j,z} + d_{j,v})^\top \mathbf{v}_{t+1}\right)\right]. \qquad (6)
\end{aligned}$$

Given that $\mathbf{s}_t$ is conditionally normal distributed, the second term on the right-hand side of the equation is

$$\begin{aligned}
\log \mathbb{E}_t \left[\exp\left((Z_s^\top d_{j,z} + d_{j,s})^\top \mathbf{s}_{t+1}\right)\right] &= (Z_s^\top d_{j,z} + d_{j,s})^\top \left[\theta_s + \Phi \mathbf{s}_t + \Phi_{s,v} \mathbf{v}_t\right] \\
&+ \frac{1}{2} (Z_s^\top d_{j,z} + d_{j,s})^\top \Sigma(\mathbf{v}_t) (Z_s^\top d_{j,z} + d_{j,s}), \qquad (7)
\end{aligned}$$

where the diagonal conditional covariance matrix $\Sigma(\mathbf{v}_t) = \Sigma^{1/2}(\mathbf{v}_t)\Sigma^{1/2}(\mathbf{v}_t)^\top$ can be written as[3]

$$\Sigma(\mathbf{v}_t) = \bar{\Sigma} + \operatorname{diag}\{\Sigma_v \mathbf{v}_t\}.$$

That is, the variance and covariance components are linear combinations of the state variables $\mathbf{v}_t$. The $N_s \times N_s$ matrix $\bar{\Sigma}$ contains the constant volatility components, and the $N_s \times N_v$ matrix Σ_v contains the sensitivity of the conditional volatilities to the state variables $\mathbf{v}_t$. It

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[3] The covariance matrix does not need to be diagonal. We assume a diagonal matrix to facilitate the exposition of the results.

follows that

$$\begin{aligned}(Z_s^\top d_{j,z} + d_{j,s})^\top \Sigma(\mathbf{v}_t)(Z_s^\top d_{j,z} + d_{j,s}) &= (Z_s^\top d_{j,z} + d_{j,s})^\top \bar{\Sigma}(Z_s^\top d_{j,z} + d_{j,s}) \\ &+ \psi(Z_s^\top d_{j,z} + d_{j,s})^\top \Sigma_v \mathbf{v}_t, \qquad (8)\end{aligned}$$

where $\psi(u) = \operatorname{diag}\{\operatorname{diag}\{u\}^2\}^\top$.

Given that $\mathbf{v}_t$ follows the multivariate autoregressive gamma process in equation (3), the third term on the right-hand side of equation (6) is

$$\log \mathbb{E}_t \left[\exp\left((Z_v^\top d_{j,z} + d_{j,v})^\top \mathbf{v}_{t+1}\right)\right] = g(Z_v^\top d_{j,z} + d_{j,v}) + h(Z_v^\top d_{j,z} + d_{j,v})^\top \mathbf{v}_t, \qquad (9)$$

where

$$g(u) = -\sum_{i=1}^{N_v} \delta_i \log(1 - u_i \varsigma_i),$$

and the i-th component of the N_v-vector $h(u)$ is $\frac{u_i \rho_i}{1 - u_i \varsigma_i}$. The scalar u_i is the i-th component of vector u.

From equations (7), (8), and (9), equation (6) can be written as

$$\begin{aligned}\bar{b}_j + (b_{j,z} - \eta_{j,z} Z_z^\top d_{j,z})^\top (\bar{z} &+ Z_z \mathbf{z}_{t-1} + Z_s \mathbf{s}_t + Z_v \mathbf{v}_t) + b_{j,zl}^\top \mathbf{z}_{t-1} + b_{j,s}^\top \mathbf{s}_t \\ + b_{j,v}^\top \mathbf{v}_t &= \eta_{j,z} \big\{ d_{j,z}^\top \bar{z} + (Z_s^\top d_{j,z} + d_{j,s})^\top \left[\theta_s + \Phi_s \mathbf{s}_t + \Phi_{s,v} \mathbf{v}_t\right] \\ &+ \frac{1}{2}(Z_s^\top d_{j,z} + d_{j,s})^\top \bar{\Sigma}(Z_s^\top d_{j,z} + d_{j,s}) + \frac{1}{2}\psi(Z_s^\top d_{j,z} + d_{j,s})^\top \Sigma_v \mathbf{v}_t \\ &+ g(Z_v^\top d_{j,z} + d_{j,v}) + h(Z_v^\top d_{j,z} + d_{j,v})^\top \mathbf{v}_t \big\}. \qquad (10)\end{aligned}$$

The coefficients $\bar{z}$, Z_z, Z_s, and Z_v have to satisfy the N_z equations of the form (10). These coefficients are found using the method of undetermined coefficients. Consider first the coefficients loading on the lagged endogenous variables $\mathbf{z}_{t-1}$. These coefficients have to satisfy

$$(b_{j,z} - \eta_{j,z} Z_z^\top d_{j,z})^\top Z_z + b_{j,zl}^\top = 0.$$

Denote by $\left[\{x_j\}_{j=1}^n\right]$ the matrix whose j-th row is the vector x_j for $j = \{1, 2, ..., n\}$. Let $B_z = \left[\{b_{j,z}^\top\}_{j=1}^{N_z}\right]_{N_z \times N_z}$, $D_z = \left[\{\eta_{j,z} d_{j,z}^\top\}_{j=1}^{N_z}\right]_{N_z \times N_z}$, $B_{zl} = \left[\{b_{j,zl}^\top\}_{j=1}^{N_z}\right]_{N_z \times N_z}$. Using this notation, the N_z equations of the above form can be written as the quadratic matrix equation

$$B_z Z_z - D_z Z_z^2 + B_{zl} = 0.$$

Solving this system provides the $N_z \times N_z$ coefficients Z_z. If the number of predetermined endogenous variables is the same as the total number of endogenous variables N_z, the solution can be found using the methods described in McCallum (1983) or Uhlig (1995). If the number of predetermined variables is lower than N_z, the solution of the quadratic matrix equation can be found numerically, under the restriction that all the elements in columns of Z_z that are loadings on non-predetermined variables are zero.

The coefficients on $\mathbf{s}_t$ in equation (10) satisfy

$$(b_{j,z} - \eta_{j,z} Z_z^\top d_{j,z})^\top Z_s + b_{j,s}^\top = \eta_{j,z}(d_{j,z}^\top Z_s + d_{j,s}^\top)\Phi_s,$$

for all j. Let $B_s = \left[\{b_{j,s}^\top\}_{j=1}^{N_z}\right]_{N_z \times N_s}$, and $D_s = \left[\{\eta_{j,z} d_{j,s}^\top\}_{j=1}^{N_z}\right]_{N_z \times N_s}$. The coefficients Z_s are found by solving the system of $N_z \times N_s$ linear equations implied by

$$(B_z - D_z Z_z) Z_s - D_z Z_s \Phi_s = D_s \Phi_s - B_s.$$

The solution for the $N_z \times N_s$ coefficients in Z_s is given by

$$\operatorname{vec}(Z_s) = \left(\mathbb{I}_{N_z \times N_s} - \Phi_s^\top \otimes (B_z - D_z Z_z)^{-1} D_z\right)^{-1} \operatorname{vec}\left((B_z - D_z Z_z)^{-1}(D_s \Phi_s - B_s)\right). \quad (11)$$

Similarly, coefficients multiplying the state variables $\mathbf{v}_t$ in equation (10) imply

$$(b_{j,z} - \eta_{j,z} Z_z^\top d_{j,z})^\top Z_v + b_{j,v}^\top = \eta_{j,z}\left[(Z_s^\top d_{j,z} + d_{j,s})^\top \Phi_{s,v} + \frac{1}{2}\psi(Z_s^\top d_{j,z} + d_{j,s})^\top \Sigma_v + h(Z_v^\top d_{j,z} + d_{j,v})^\top\right].$$

Let

$$\begin{aligned}
B_v &= \left[\{b_{j,v}^\top\}_{j=1}^{N_z}\right]_{N_z \times N_v}, \\
D_v &= \left[\{\eta_{j,z} d_{j,v}^\top\}_{j=1}^{N_z}\right]_{N_z \times N_v}, \\
\Psi_v(\boldsymbol{\eta_z}, Z_s, D_z, D_s) &= \left[\{\eta_{j,z}\psi(Z_s^\top d_{j,z} + d_{j,s})^\top\}_{j=1}^{N_z}\right]_{N_z \times N_v}, \\
H(\boldsymbol{\eta_z}, Z_v, D_z, D_v) &= \left[\{\eta_{j,z} h(Z_v^\top d_{j,z} + d_{j,v})^\top\}_{j=1}^{N_z}\right]_{N_z \times N_v},
\end{aligned}$$

where $\boldsymbol{\eta_z} = (\eta_{1,z}, \eta_{2,z}, ..., \eta_{N_z,z})^\top$. Using this notation, the coefficients Z_v are found by solving the system of $N_z \times N_v$ quadratic equations implied by

$$(B_z - D_z Z_z) Z_v + B_v = (D_z Z_s + D_s)\Phi_{s,v} + \frac{1}{2}\Psi_v(\boldsymbol{\eta_z}, Z_s, D_z, D_s)\Sigma_v + H(\boldsymbol{\eta_z}, Z_v, D_z, D_v).$$

Finally, the constant coefficients in equation (10) imply

$$\begin{aligned}
\bar{b}_j + (b_{j,z} - \eta_{j,z} Z_z^\top d_{j,z})^\top \bar{z} &= \eta_{j,z}\Big[d_{j,z}^\top \bar{z} + (Z_s^\top d_{j,z} + d_{j,s})^\top \theta_s \\
&+ \frac{1}{2}(Z_s^\top d_{j,z} + d_{j,s})^\top \bar{\Sigma}(Z_s^\top d_{j,z} + d_{j,s}) + g(Z_v^\top d_{j,z} + d_{j,v})\Big].
\end{aligned}$$

Let

$$\begin{aligned} \bar{B} &= \left[\{\bar{b}_j\}_{j=1}^{N_z}\right]_{N_z \times 1}, \\ G(\boldsymbol{\eta_z}, Z_v, D_z, D_v) &= \left[\{\eta_{j,z} g(Z_v^\top d_{j,z} + d_{j,v})^\top\}_{j=1}^{N_z}\right]_{N_z \times 1}, \\ \text{and} \quad \bar{\Psi}(\boldsymbol{\eta_z}, Z_s, D_z, D_s) &= \left[\{\eta_{j,z} (Z_s^\top d_{j,z} + d_{j,s})^\top \bar{\Sigma} (Z_s^\top d_{j,z} + d_{j,s})^\top\}_{j=1}^{N_z}\right]_{N_z \times 1}. \end{aligned}$$

The coefficients $\bar{z}$ are found by solving the system of N_z linear equations implied by

$$(B_z - D_z Z_z - D_z)\bar{z} + \bar{B} = (D_z Z_s + D_s)\theta_s + \frac{1}{2}\bar{\Psi}(\boldsymbol{\eta_z}, Z_s, D_z, D_s) + G(\boldsymbol{\eta_z}, Z_v, D_z, D_v).$$

Linearization points

The expectational equations (6) are frequently obtained from log-linearizations of the model around linearization points of the variables under study. The accuracy of the approximation depends on these points. A reasonable candidate for a linearization point is given by the unconditional expectation of the respective variable. Since this unconditional expectation is endogenous, the solution for the linearization point involves a fixed point problem. Consider N_m linearization points $\bar{m}_k$, for $k = \{1, 2, ..., N_m\}$, described by the unconditional expectation

$$\begin{aligned} \bar{m}_k &= \mathbb{E}\Big[\bar{a}_k + a_{k,z}^\top \mathbf{z}_t + a_{k,zl}^\top \mathbf{z}_{t-1} + a_{k,s}^\top \mathbf{s}_t + a_{k,v}^\top \mathbf{v}_t \\ &+ \eta_{k,m} \log \mathbb{E}_t \left[\exp(c_{k,z}^\top \mathbf{z}_{t+1} + c_{k,s}^\top \mathbf{s}_{t+1} + c_{k,v}^\top \mathbf{v}_{t+1})\right]\Big], \end{aligned} \qquad (12)$$

where all scalars and vectors multiplying the model variables have appropriate dimensions.

Given the model solution, the linearization point becomes

$$\begin{aligned}
\bar{m}_k &= \mathbb{E}\Big[\bar{a}_k + (a_{k,z} + \eta_{k,m} Z_z^\top c_{k,z})^\top \mathbf{z}_t + a_{k,zl}^\top \mathbf{z}_{t-1} + a_{k,s}^\top \mathbf{s}_t + a_{k,v}^\top \mathbf{v}_t + \eta_{k,m} c_{k,z}^\top \bar{z} \\
&+ \eta_{k,m}(Z_s^\top c_{k,z} + c_{k,s})^\top \mathbb{E}_t[\mathbf{s}_{t+1}] + \frac{1}{2}\eta_{k,m}(Z_s^\top c_{k,z} + c_{k,s})^\top(\bar{\Sigma} + \mathrm{diag}\{\Sigma_v \mathbf{v}_t\})(Z_s^\top c_{k,z} + c_{k,s}) \\
&+ \eta_{k,m} g(Z_v^\top c_{k,z} + c_{k,v}) + \eta_{k,m} h(Z_v^\top c_{k,z} + c_{k,v})^\top \mathbf{v}_t\Big] \\
&= \bar{a}_k + (a_{k,z} + \eta_{k,m} Z_z^\top c_{k,z} + a_{k,zl})^\top \mathbb{E}[\mathbf{z}_t] + \eta_{k,m} c_{k,z}^\top \bar{z} \\
&+ \frac{1}{2}\eta_{k,m}(Z_s^\top c_{k,z} + c_{k,s})^\top \bar{\Sigma}(Z_s^\top c_{k,z} + c_{k,s}) \\
&+ \eta_{k,m} g(Z_v^\top c_{k,z} + c_{k,v}) + \big(a_{k,s} + \eta_{k,m}(Z_s^\top c_{k,z} + c_{k,s})\big)^\top \mathbb{E}[\mathbf{s}_t] \\
&+ \left(a_{k,v} + \frac{1}{2}\eta_{k,m}\mathrm{diag}\{\mathrm{diag}\{Z_s^\top c_{k,z} + c_{k,s}\}^2\}\Sigma_v + \eta_{k,m} h(Z_v^\top c_{k,z} + c_{k,v})\right)^\top \mathbb{E}[\mathbf{v}_t],
\end{aligned}$$

where $\mathbb{E}[\mathbf{s}_t] = (\mathbb{I}_{N_s} - \Phi_s)^{-1}(\theta_s + \Phi_{s,v}\mathbb{E}_t[\mathbf{v}_t])$, $\mathbb{E}[\mathbf{v}_t]$ has components $\mathbb{E}[\mathbf{v}_{i,t}] = \frac{c_i \delta_i}{1-\rho_i}$ for $i = 1, ..., N_v$, and $\mathbb{E}[\mathbf{z}_t] = (\mathbb{I}_{N_z} - Z_z)^{-1}(\bar{z} + Z_s \mathbb{E}[\mathbf{s}_t] + Z_v \mathbb{E}[\mathbf{v}_t])$. In matrix form, $\bar{M} = (\bar{m}_1, \bar{m}_2, ..., \bar{m}_{N_m})^\top$ is given by

$$\begin{aligned}
\bar{M} &= \bar{A} + (A_z + C_z Z_z + A_{zl})\mathbb{E}[\mathbf{z}_t] + C_z \bar{z} + \frac{1}{2}\bar{\Psi}(\boldsymbol{\eta_m}, Z_s, C_z, C_s) \\
&+ G(\boldsymbol{\eta_m}, Z_v, C_z, C_v) + (A_s + C_z Z_s + C_s)\,\mathbb{E}[\mathbf{s}_t] \\
&+ \left(A_v + \frac{1}{2}\Psi_v(\boldsymbol{\eta_m}, Z_s, C_z, C_s) + H(\boldsymbol{\eta_m}, Z_v, C_z, C_v)\right)\mathbb{E}[\mathbf{v}_t],
\end{aligned}$$

where

$$
\begin{aligned}
\bar{A} &= \left[\{\bar{a}_k\}_{k=1}^{N_m}\right]_{N_m \times 1}, \\
A_z &= \left[\{a_{k,z}^\top\}_{k=1}^{N_m}\right]_{N_m \times N_z}, \\
A_{zl} &= \left[\{a_{k,zl}^\top\}_{k=1}^{N_m}\right]_{N_m \times N_z}, \\
A_s &= \left[\{a_{k,s}^\top\}_{k=1}^{N_m}\right]_{N_m \times N_s}, \\
A_v &= \left[\{a_{k,v}^\top\}_{k=1}^{N_m}\right]_{N_m \times N_v}, \\
C_z &= \left[\{\eta_{k,m} c_{k,z}^\top\}_{k=1}^{N_m}\right]_{N_m \times N_z}, \\
C_s &= \left[\{\eta_{k,m} c_{k,s}^\top\}_{k=1}^{N_m}\right]_{N_m \times N_s}, \\
C_v &= \left[\{\eta_{k,m} c_{k,v}^\top\}_{k=1}^{N_m}\right]_{N_m \times N_v}, \\
\bar{\Psi}(\boldsymbol{\eta_m}, Z_s, C_z, C_s) &= \left[\{\eta_{k,m}(Z_s^\top c_{k,z} + c_{k,s})^\top \bar{\Sigma}(Z_s^\top c_{k,z} + c_{k,s})^\top\}_{k=1}^{N_m}\right]_{N_m \times 1}, \\
\Psi_v(\boldsymbol{\eta_m}, Z_s, C_z, C_s) &= \left[\{\eta_{k,m}\psi(Z_s^\top c_{k,z} + c_{k,s})^\top\}_{k=1}^{N_m}\right]_{N_m \times N_v}, \\
G(\boldsymbol{\eta_m}, Z_v, C_z, C_v) &= \left[\{\eta_{k,m} g(Z_v^\top c_{k,z} + c_{k,v})^\top\}_{k=1}^{N_m}\right]_{N_m \times 1}, \\
H(\boldsymbol{\eta_m}, Z_v, C_z, C_v) &= \left[\{\eta_{k,m} h(Z_v^\top c_{k,z} + c_{k,v})^\top\}_{k=1}^{N_m}\right]_{N_m \times N_v},
\end{aligned}
$$

for $\boldsymbol{\eta_m} = (\eta_{1,m}, \eta_{2,m}, ..., \eta_{N_m,m})^\top$. For an initial guess for all $\bar{m}_k$'s affecting equations (6), the system of equations is solved, and a new set $\bar{M}$ is found. The procedure is repeated until convergence. That is, until $M^{(n+1)} \approx M^{(n)}$. Given the particular nature and parameter values of the model, it is possible that a fixed point cannot be found. In this case, the iteration procedure can be stopped at iteration n if $\|M^{(n+1)} - M^{(n)}\| > \|M^{(n)} - M^{(n-1)}\|$. That is, if the divergence between previous and new linearization points increases.

3 Applications

We present here two economic applications of the solution method. The first application is the endowmnent economy of Bansal and Yaron (2004) that is used to analyze asset pricing dynamics. Time-varying economic uncertainty plays an important role in this economy. Asset holders require expected excess returns that reflect a compensation for this risk (volatility premium). The second application is a simple New Keynesian model where policy shocks have stochastic volatility. Time-varying volatility can be helpful to understand macroeconomic dynamics as shown by Justiniano and Primiceri (2008) and **?**. We characterize the model dynamics in term of the solution method above and find the model solutions for the two applications. Since the two models involve log-linearizations, we provide an analysis of the accuracy of the solutions.

3.1 Bansal and Yaron (2004) Asset Pricing Model

Consider a representative agent maximizing utility

$$W_t = \left((1-\beta) C_t^{1-1/\psi} + \beta \mathbb{E}_t \left[W_{t+1}^{1-\gamma} \right]^{\frac{1-1/\psi}{1-\gamma}} \right)^{\frac{1}{1-1/\psi}},$$

where β is the time preference parameter, ψ is the elasticity of intertemporal substitution, and γ is the degree of relative risk aversion. The equation can be written in terms of utility scaled by consumption as

$$\frac{W_t}{C_t} = \left(1 - \beta + \beta \mathbb{E}_t \left[\left(\frac{W_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \right)^{1-\gamma} \right]^{\frac{1-1/\psi}{1-\gamma}} \right)^{\frac{1}{1-1/\psi}}.$$

Denote consumption growth by $\Delta c_t = \log C_t - \log C_{t-1}$, and the utility scaled by consumption by $w_t = \log W_t - \log C_t$, to obtain

$$\left(1 - \frac{1}{\psi}\right) w_t = \log\left(1 - \beta + \beta \mathbb{E}_t \left[\exp\left((1-\gamma)(w_{t+1} + \Delta c_{t+1})\right)\right]^{\frac{1-1/\psi}{1-\gamma}}\right). \tag{13}$$

This equation can be log-linearized around the stochastic steady state

$$\bar{m}_w = \mathbb{E}\left\{\log \mathbb{E}_t \left[\exp\left((1-\gamma)(w_{t+1} + \Delta c_{t+1})\right)\right]\right\}, \tag{14}$$

to obtain

$$\left(1 - \frac{1}{\psi}\right) w_t = \bar{\eta}_w + \eta_w \log \mathbb{E}_t \left[\exp\left((1-\gamma)(w_{t+1} + \Delta c_{t+1})\right)\right], \tag{15}$$

where

$$\eta_w = \frac{\beta}{\mu_w}\left(\frac{1-1/\psi}{1-\gamma}\right)\exp\left[\left(\frac{1-1/\psi}{1-\gamma}\right)\bar{m}_w\right], \quad \bar{\eta}_w = \log \mu_w - \bar{m}_w \eta_w,$$

and

$$\mu_w = 1 - \beta + \beta \exp\left[\left(\frac{1-1/\psi}{1-\gamma}\right)\bar{m}_w\right].$$

The stochastic discount factor for this economy is

$$M_{t,t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-1/\psi}\left(\frac{W_{t+1}}{\mathbb{E}_t[W_{t+1}^{1-\gamma}]^{1/(1-\gamma)}}\right)^{1/\psi-\gamma}. \tag{16}$$

Replacing equation (13) in equation (16), the log -pricing kernel $m_{t,t+1} \equiv \log M_{t,t+1}$ becomes

$$m_{t,t+1} = \log\beta + \left(\frac{1/\psi-\gamma}{1-\gamma}\right)\frac{\bar{\eta}_w}{\eta_w} - \gamma\Delta c_{t+1} - \left(\frac{1/\psi-\gamma}{1-\gamma}\right)\left(\frac{1-1/\psi}{\eta_w}\right) w_t + \left(\frac{1}{\psi} - \gamma\right) w_{t+1}.$$

The exogenous dynamics of consumption growth, Δc_t, and dividend growth, Δd_t, are modeled as

$$\begin{aligned}\Delta c_{t+1} &= \mu_c + x_t + \sigma_{c,t}\varepsilon_{c,t+1},\\ x_{t+1} &= \phi_x x_t + \sigma_{x,t}\varepsilon_{x,t+1},\\ \Delta d_{t+1} &= \mu_d + \phi_{dx} x_t + \sigma_{d,t}\varepsilon_{d,t+1},\end{aligned}$$

where x_t is the time-varying component of expected consumption growth. The innovations ε_c, ε_x, and ε_d are i.i.d. standard normal and uncorrelated among them. Conditional volatilities are modeled as

$$\sigma_{j,t} \equiv \sigma_j (1 - I_v + I_v v_t)^{1/2}, \tag{17}$$

for $j = \{c, x, d\}$, where v_t is an autoregressive gamma process with parameters ς_v, δ_v, and ρ_v. The indicator function I_v allows us to analyze economies with homoscedastic ($I_v = 0$) and heteroscedastic ($I_v = 1$) shocks. The specification for time-varying volatility differs from the approximated discrete square root process in Bansal and Yaron (2004). A comparable specification would be

$$v_{t+1} = (1 - \phi_v)\theta_v + \phi_v v_t + \sigma_v v_t^{1/2} \varepsilon_{v,t+1},$$

where $\varepsilon_{v,t} \sim \mathrm{IID}\mathcal{N}(0,1)$.[4] A limitation of the approximate discrete square root process is the possibility for the variance v_t to take negative values. This possibility is ruled out for autoregressive gamma processes.

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[4] Strictly speaking, the volatility process in Bansal and Yaron exhibits constant conditional volatility. We allow for time-varying volatility in volatility in the approximate square root process to make it comparable to the autoregressive gamma process.

The stock price is $S_t = \mathbb{E}_t[M_{t,t+1}(D_{t+1} + S_{t+1})]$, and, therefore, the log price-dividend ratio $p_t = \log S_t - \log D_t$ satisfies

$$p_t = \log \mathbb{E}_t \left[\exp \left(m_{t,t+1} + \Delta d_{t+1} + \log \left(1 + e^{p_{t+1}} \right) \right) \right]. \tag{18}$$

This equation can be log-linearized around

$$\bar{m}_p = \mathbb{E} \left\{ \log \mathbb{E}_t \left[\exp \left(m_{t,t+1} + \Delta d_{t+1} + \bar{\eta}_p + \eta_p p_{t+1} \right) \right] \right\}, \tag{19}$$

where

$$\eta_p = \frac{e^{\bar{m}_p}}{1 + e^{\bar{m}_p}}, \quad \text{and} \quad \bar{\eta}_p = \log \left(1 + e^{\bar{m}_p} \right) - \bar{m}_p \eta_p.$$

The approximate equation for the price-dividend ratio becomes

$$p_t = \log \mathbb{E}_t \left[\exp \left(m_{t,t+1} + \Delta d_{t+1} + \bar{\eta}_p + \eta_p p_{t+1} \right) \right]. \tag{20}$$

Notice that the one-period stock return $r_{m,t+1}$ is given by

$$e^{r_{m,t+1}} = \left(1 + \frac{S_{t+1}}{D_{t+1}} \right) \left(\frac{D_{t+1}}{D_t} \right) \left(\frac{D_t}{S_t} \right),$$

and the approximation for the price-dividend ratio implies

$$r_{m,t+1} = \bar{\eta}_p + \eta_p p_{t+1} + \Delta d_{t+1} - p_t. \tag{21}$$

The risk-free rate is

$$r_t = - \log \mathbb{E}_t [\exp(m_{t,t+1})], \tag{22}$$

and the equity premium is

$$\log\left(\frac{\mathbb{E}_t[\exp(r_{m,t+1})]}{\exp(r_t)}\right) = \mathbb{E}_t[r_{m,t+1} - r_t] + J.I._{\cdot t},$$

where $J.I._{\cdot t}$ is a Jensen's inequality term characterized in Appendix A.

The system of expectational equations is described by equations (15), (20), and (22). Notice that these equations conform to the functional form (4). The system can be described in terms of the solution method in Section 2 by the state variables $\mathbf{s}_t = (\Delta c_t, x_t, \Delta d_t)^\top$ and $\mathbf{v}_t = v_t$, and the endogenous variables $\mathbf{z}_t = (w_t, p_t, r_t)^\top$.[5] Equations (14) and (19) for the endogenous linearization points have the functional form (12). The vectors and matrices to find the solution become

$$\bar{B} = \begin{bmatrix} -\bar{\eta}_w \\ -\log\beta - \bar{\eta}_p - \left(\frac{1/\psi-\gamma}{1-\gamma}\right)\frac{\bar{\eta}_v}{\eta_v} \\ -\log\beta - \left(\frac{1/\psi-\gamma}{1-\gamma}\right)\frac{\bar{\eta}_v}{\eta_v} \end{bmatrix}, \qquad B_z = \begin{bmatrix} 1-1/\psi & 0 & 0 \\ \left(\frac{1/\psi-\gamma}{1-\gamma}\right)\left(\frac{1-1/\psi}{\eta_v}\right) & 1 & 0 \\ \left(\frac{1/\psi-\gamma}{1-\gamma}\right)\left(\frac{1-1/\psi}{\eta_v}\right) & 0 & -1 \end{bmatrix},$$

$$B_{zl} = B_s = \mathbf{0}_{3\times 3}, \qquad B_v = \mathbf{0}_{3\times 1},$$

$$D_z = \begin{bmatrix} \eta_w(1-\gamma) & 0 & 0 \\ 1/\psi - \gamma & \eta_p & 0 \\ 1/\psi - \gamma & 0 & 0 \end{bmatrix}, \quad \text{and} \quad D_s = \begin{bmatrix} \eta_w(1-\gamma) & 0 & 0 \\ -\gamma & 0 & 1 \\ -\gamma & 0 & 0 \end{bmatrix},$$

$D_v = \mathbf{0}_{3\times 1}$, $\bar{\Sigma} = (1 - I_v)\text{diag}\{\sigma_c^2, \sigma_x^2, \sigma_d^2\}$, and $\Sigma_v = I_v(\sigma_c^2, \sigma_x^2, \sigma_d^2)^\top$.

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[5] In fact, the risk-free rate and its corresponding equation (22) do not need to be included in the system to find the solution. However, including them allows us to compute easily the sensititivity of the risk-free rate to the state variables and shocks in the economy. A similar exercise can be done including the stock return as an endogenous variable and adding equation (21) to the system.

The matrices for the linearization points $\bar{M} = (\bar{m}_w, \bar{m}_p)^\top$ are

$$\bar{A} = \begin{bmatrix} 0 \\ \log\beta + \bar{\eta}_p + \left(\frac{1/\psi-\gamma}{1-\gamma}\right)\frac{\bar{\eta}_v}{\eta_v} \end{bmatrix}, \quad A_z = \begin{bmatrix} 0 & 0 & 0 \\ -\left(\frac{1/\psi-\gamma}{1-\gamma}\right)\frac{1-1/\psi}{\eta_v} & 0 & 0 \end{bmatrix}, \quad A_{zl} = A_s = \mathbf{0}_{2\times 3},$$

$$A_v = \mathbf{0}_{2\times 1}, \quad C_z = \begin{bmatrix} 1-\gamma & 0 & 0 \\ 1/\psi-\gamma & \eta_p & 0 \end{bmatrix}, \quad \text{and} \quad C_s = \begin{bmatrix} 1-\gamma & 0 & 0 \\ -\gamma & 0 & 1 \end{bmatrix}.$$

The baseline parameter values are presented in Table 1. These parameters are comparable to those under the preferred specification of Bansal and Yaron (2004). The parameter values for the stochastic volatility process deserve further explanation. The autocorrelation parameter ρ_v is set equal to the autocorrelation parameter for the variance process in Bansal and Yaron (2004). The parameters ς_v and δ_v are chosen such that the unconditional mean and variance of the consumption process are the same as in Bansal and Yaron (2004). Tables 2 and 3 show the model-implied statistics for macroeconomic and financial variables. The quantitative results are very similar to those in Bansal and Yaron (2004).

Volatility Premium

We compare the volatility premium in the Bansal and Yaron specification and the volatility premium using the autoregressive gamma process. The volatility premium can be obtained from the price of volatility risk in the log-pricing kernel and the sensitivity of stock returns to this risk. Specifically, the log-pricing kernel can be written as

$$m_{t,t+1} = \mathbb{E}_t[m_{t,t+1}] - \lambda_c\sigma_{c,t}\varepsilon_{c,t+1} - \lambda_x\sigma_{x,t}\varepsilon_{x,t+1} - \lambda_v(v_{t+1} - \mathbb{E}_t[v_{t+1}]),$$

where λ_v is the price of volatility risk, ant the stock return is

$$r_{m,t+1} = \mathbb{E}_t[r_{m,t+1}] + r_c \sigma_{c,t} \varepsilon_{c,t+1} + r_x \sigma_{x,t} \varepsilon_{x,t+1} + r_v (v_{t+1} - \mathbb{E}_t[v_{t+1}]),$$

where r_v is the sensitivity of the stock return to volatility. Under the Bansal and Yaron specification in equation (17), the volatility premium is $\lambda_v r_v \sigma_v^2 v_t$. Under the autoregressive gamma specification, Appendix A shows that the volatility premium is

$$\delta_v \log\left[\frac{1 + (\lambda_v - r_v)\varsigma_v}{(1 + \lambda_v \varsigma_v)(1 - r_v \varsigma_v)}\right] - \rho_v \varsigma_v \left(\frac{(\lambda_v - r_v)^2}{1 + (\lambda_v - r_v)\varsigma_v} - \frac{\lambda_{v,}^2}{1 + \lambda_v \varsigma_v} + \frac{r_v^2}{1 - r_v \varsigma_v}\right) v_t.$$

To have a valid comparison, we need similar properties for the two volatility processes. Specifically, the autocorrelation of the two volatility processes is the same if the persistence coefficients ϕ_v and ρ_v are the same. The unconditional mean and variance of volatility are the same by making $\theta_v = 1$, $\varsigma_v = \frac{\sigma_v^2}{1+\phi_v}$, and $\delta_v = \frac{\theta_v(1-\phi_v)}{\varsigma_v}$, for some value for σ_v. Figure 1 shows the volatility premia for different values for ϕ_v and σ_v under the two specifications. The figure shows that the volatility premium is lower under the autoregressive gamma specification. The differences between the two premia increases as the persistence and conditional volatility of the volatility processes increase. The difference is the result of the positive probability assigned under the Bansal and Yaron specification to negative outcomes of volatility, which increases the negative covariance between the pricing kernel and stock returns. The Bansal and Yaron calibration implies $\phi_v = 0.987$ and $\sigma_v = 0.0378$ which translates into a small difference of 5 bps in volatility premia across specifications. However, the difference can increase to 70 bps if $\sigma_v = 0.09$, or 30 bps if ϕ_v =0.995.

Solution Method Accuracy

We analyze the effect of time-varying volatility on the accuracy of the solution method using descriptive statistics and a formal statistical test. Consider the errors implied by the solution for the two model equations that are linearized: the recursive equation for the normalized utility (13), and the return equation for price-dividend ratio (18). The errors are given by

$$\begin{aligned} e_{w,t} &= \log\left(1-\beta+\beta\mathbb{E}_t\left[\exp\left((1-\gamma)(w_{t+1}+\Delta c_{t+1})\right)\right]^{\frac{1-1/\psi}{1-\gamma}}\right)-\left(1-\frac{1}{\psi}\right)w_t, \\ \text{and} \quad e_{p,t} &= \log\mathbb{E}_t\left[\exp\left(m_{t,t+1}+\Delta d_{t+1}+\log\left(1+e^{p_{t+1}}\right)\right)\right]-p_t. \end{aligned} \tag{23}$$

We simulate a sample of 10,000 observations for the exogenous processes Δc_t, x_t, and Δd_t, and compute the errors $e_{w,t}$ and $e_{p,t}$. The expectation in the error equation $e_{w,t}$ has an analytical solution given the linearity of the solution for w_t. The expectation in the error equation $e_{p,t}$ is found numerically using 10,000 simulations for all processes at $t+1$ for each simulated value of the processes at t. Table 4 shows descriptive statistics of the errors implied by the approximate solution for the simulated data. For the model with constant volatility ($I_v = 0$), the errors $e_{w,t}$ have a mean of 3.28×10^{-7} and a standard deviation of 4.54×10^{-7}. These values are very small in comparison to, for instance, the value of 0.12 for the unconditional mean of w_t. The errors $e_{p,t}$ have a mean of 3.14×10^{-5} and a standard deviation of 1.42×10^{-3}. These values are also very small in comparison to the value of 5.70 for the unconditional mean of p_t. For the model with stochastic volatility, the errors $e_{w,t}$ have a mean of 3.93×10^{-7} and a standard deviation of 5.35×10^{-7}. The value for the unconditional mean of w_t is 0.07. The errors $e_{p,t}$ have a mean of 4.77×10^{-5} and a standard deviation of 1.46×10^{-3}. The value for the unconditional mean of p_t is 5.68. In summary, this error analysis suggests that the accuracy of the solution is not significantly negatively affected by time-varying volatility.

More formally, consider the Den Haan and Marcet (1994) accuracy test, summarized in Appendix C. We test the hypothesis that the errors of the model solution are zero. We consider the errors implied by the three equations (13), (18), and (22), and use two instruments: a constant, and x_t. Therefore, the test involves a chi-square distribution with $3 \times 2 = 6$ degrees of freedom. Table 5 presents the test results. Panels A, B, and C show the rejection rates in the upper and lower critical regions of a $\chi^2(6)$ distribution for models with different values for γ, ψ, and ϕ_x, respectively.[6] The values with and without parentheses denote rejection rates for specifications with and without time-varying volatility, respectively. Consider first the results for models with time-varying volatility. Using benchmark parameter values, the solution obtained through the log-linear approximation is rejected 8.4% of the time in the lower 5% critical region while the rejection rate is only 0.3% in the upper 5% critical region. As γ is lowered to 7.5, the probability of rejection increases drastically to 27.1% in the lower tail. Furthermore, when ψ is high at 0.99, the rejection rates increase to 30.1% and 2.2% in the lower and upper regions, respectively. Finally, panel C shows that the model is not very sensitive to variations in ϕ_x as the rejection rates in both regions are largely unchanged. Consider now the results for models with constant volatility. For the benchmark parameter values, the solution is more likely to be rejected than in the model with time-varying volatility. The rejection rate in the lower critical region increases from 8.4% to 11.6%, and it increases from 0.3% to 0.6% in the upper critical region. Similar conclusions can be obtained from the other parameterizations. Overall, while the accuracy of the solution method depends significantly on the parameter values, time-varying volatility does not negatively affect this accuracy.

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[6] If these values are less than 5%, the hypothesis of zero errors cannot be rejected.

3.2 A New Keynesian Model with Stochastic Volatility

Consider a simple New Keynesian model with price rigidities and policy shocks with time-varying volatility. We analyze the difference in the implied dynamics of inflation and the output gap between a linearized solution and the solution proposed here. We provide an analysis of the effects of time-varying volatility on the accuracy of the two solutions.

For simplicity we assume that the efficient output of the economy is constant. Therefore, the only source of variability in inflation and output is policy shocks. The equations characterizing optimality in the economy are given by

$$
\begin{aligned}
e^{-i_t} &= \mathbb{E}_t[M_{t,t+1}], && (24)\\
\left[\frac{1}{1-\alpha}\left(1-\alpha\Pi_t^{-(1-\theta)}\right)\right]^{\frac{1+\theta\omega}{1-\theta}} H_t &= X_t^{\omega+\gamma} G_t, && (25)\\
i_t &= \bar{\imath} + \imath_\pi \pi_t + u_t. && (26)
\end{aligned}
$$

Equation (24) is the optimality condition for households, linking the short-term nominal interest rate, i_t to the marginal rate of substitution of nominal consumption $M_{t,t+1}$. We assume power utility with coefficient of relative risk aversion given by γ, to obtain a marginal rate of substitution given by

$$\log M_{t,t+1} = \log\beta - \gamma\Delta x_{t+1} - \pi_{t+1},$$

where $x_t \equiv \log X_t$ is the output gap, and $\pi_t \equiv \log \Pi_t$ is the inflation rate for the economy. The parameter β is the subjective discount factor, and Δ is the difference operator. Equation (25) is the optimality condition for the production sector. It connects the optimal price under Calvo (1983) price rigidities to the marginal cost of the firm and the markup. The parameter α is the probability of adjusting the price optimally at a particular time. The elasticity of substitution across goods is θ, and the Frisch elasticity of labor supply is $1/\omega$. The appendix shows that the processes H_t and G_t are characterized recursively as

$$\begin{aligned} H_t &= 1 + \alpha \mathbb{E}_t \left[M_{t,t+1} \left(\frac{X_{t+1}}{X_t} \right) \Pi_{t+1}^{\theta} H_{t+1} \right], \qquad (27) \\ \text{and} \quad G_t &= 1 + \alpha \mathbb{E}_t \left[M_{t,t+1} \left(\frac{X_{t+1}}{X_t} \right)^{1+\omega+\gamma} \Pi_{t+1}^{\theta(1+\omega)+1} G_{t+1} \right]. \qquad (28) \end{aligned}$$

Equation (26) is the policy rule. The rule is affected by policy shocks u_t following the process

$$u_{t+1} = \phi_u u_t + \sigma_u \left(1 - q + q v_t\right)^{1/2} \varepsilon_{u,t+1},$$

where v_t follows an autoregressive gamma process with parameters (c_v, ρ_v, δ_v). The parameter q allows us to make comparisons of model solutions from the case of homoskedastic shocks when $q = 0$, to the case where all the volatility in shocks is determined by the time-varying component v_t when $q = 1$. The unconditional volatility of policy shocks is the same across model comparisons by making $\mathbb{E}[v_t] = \frac{c_v \delta_v}{1-\rho_v} = 1$. For a given value of the autocorrelation coefficient ρ_v, the coefficients δ_v and c_v are obtained from setting the volatility to expected value ratio to a specific value. It provides $\sqrt{\delta} = \left(\frac{\sigma(v_t)}{\mathbb{E}[v_t]} \right)^{-1}$, and $c_v = \frac{1-\rho_v}{\delta_v}$.

Traditionally, the solution of the New Keynesian model is found based on first-order approximations where the optimality conditions for households and the productive sector are linearized as

$$\begin{aligned} i_t &= -\log \beta + \gamma \mathbb{E}_t[\Delta x_{t+1}] + \mathbb{E}_t[\pi_{t+1}], \\ \pi_t &= \kappa x_t + \beta \mathbb{E}_t[\pi_{t+1}], \end{aligned}$$

respectively, for $\kappa = \frac{(1-\alpha)(1-\alpha\beta)(\omega+\gamma)}{\alpha(1+\theta\omega)}$. It is easy to show, using the method of undetermined coefficients, that solutions for inflation and the output gap are given by

$$x_t = \bar{x} + x_u u_t, \quad \text{and} \quad \pi_t = \bar{\pi} + \pi_u u_t,$$

where

$$\bar{x} = -\frac{1-\beta}{\kappa}\frac{\log\beta + \bar{\imath}}{\imath_\pi - 1}, \quad x_u = -\frac{1-\beta\phi_u}{\kappa(\imath_\pi - \phi_u) + \gamma(1-\phi_u)(1-\beta\phi_u)},$$

$$\bar{\pi} = -\frac{\log\beta + \bar{\imath}}{\imath_\pi - 1}, \quad \text{and} \quad \pi_u = -\frac{\kappa}{\kappa(\imath_\pi - \phi_u) + \gamma(1-\phi_u)(1-\beta\phi_u)}.$$

Notice that stochastic volatility does not play a role in the equilibrium implied by the first-order approximation.

Alternatively, the recursive equations (27) and (28) can be log-linearized as

$$\begin{aligned} h_t &= \bar{\eta}_h + \eta_h \mathbb{E}_t[e^{(1-\gamma)\Delta x_{t+1} + (\theta-1)\pi_{t+1} + h_{t+1}}], && (29) \\ \text{and} \quad g_t &= \bar{\eta}_g + \eta_g \mathbb{E}_t[e^{(1+\omega)\Delta x_{t+1} + \theta(1+\omega)\pi_{t+1} + g_{t+1}}], && (30) \end{aligned}$$

respectively, where $h_t \equiv \log H_t$, $g_t \equiv \log G_t$, and the linearization coefficients are $\eta_j = \frac{e^{\alpha\beta\bar{m}_j}}{1+e^{\alpha\beta\bar{m}_j}}$, $\bar{\eta}_j = \log\left(1 + e^{\alpha\beta\bar{m}_j}\right) - \eta_j \bar{m}_j$, for linearization points $\bar{m}_j$ to be determined for $j = \{h, g\}$. The system of log-linearized equations is completed with the log-linearization of equation (25) which yields

$$\bar{\eta}_\pi + \eta_h \pi_t + h_t = (\omega + \gamma)x_t + g_t,$$

where $\eta_\pi = (1+\theta\omega)\frac{\alpha e^{-(1-\theta)\pi^*}}{1+\alpha e^{-(1-\theta)\pi^*}}$, and $\bar{\eta}_\pi = \frac{1+\theta\omega}{1-\theta}\log\left(\frac{1-\alpha e^{-(1-\theta)\pi^*}}{1-\alpha}\right) - \eta_\pi \pi^*$.

The solution of this system implies linear solutions for x_t, π_t, h_t, and g_t that depend on policy shocks and the volatility of the shocks. Notice that equations (29) and (30) have the form of the expectational equation (4). The solution of the system can then be obtained using the method in Section 2. In particular, let $\mathbf{z}_t = (x_t, \pi_t, h_t, g_t)^\top$, $\mathbf{s}_t = u_t$, and $\mathbf{v}_t = v_t$. The vector and matrices in Section 2 become

$$\bar{B} = \mathbf{0}_{4\times 1}, \quad B_z = \begin{bmatrix} -(\omega+\gamma) & \eta_\pi & 1 & -1 \\ \eta_h(1-\gamma) & 0 & 1 & 0 \\ \eta_g(1+\omega) & 0 & 0 & 1 \\ -\gamma & -\imath_\pi & 0 & 0 \end{bmatrix}, \quad B_{zl} = \mathbf{0}_{4\times 1} \quad B_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad B_v = \mathbf{0}_{4\times 1}$$

$$D_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \eta_h(1-\gamma) & \eta_h(\theta-1) & \eta_h & 0 \\ \eta_g(1+\omega) & \eta_g\theta(1+\omega) & 0 & \eta_g \\ -\gamma & -1 & 0 & 0 \end{bmatrix}, \quad D_s = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad D_v = \mathbf{0}_{4\times 1},$$

$\bar{\Sigma} = (1-q)\sigma_u^2$, and $\Sigma_v = q\sigma_u^2$.

The linearization points are the unconditional expectations

$$\begin{aligned} \pi^* &= \mathbb{E}[\pi_t], \\ \bar{m}_h &= \mathbb{E}\left[\log \mathbb{E}_t[e^{(1-\gamma)\Delta x_{t+1} + (\theta-1)\pi_{t+1} + h_{t+1}}]\right], \\ \bar{m}_g &= \mathbb{E}\left[\log \mathbb{E}_t[e^{(1+\omega)\Delta x_{t+1} + \theta(1+\omega)\pi_{t+1} + g_{t+1}}]\right]. \end{aligned}$$

These equations have the form of equation (12). The vectors and matrices associated to the linearization points become

$$\bar{A} = \mathbf{0}_{3\times 1}, \quad A_z = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(1-\gamma) & 0 & 0 & 0 \\ -(1+\omega) & 0 & 0 & 0 \end{bmatrix}, \quad A_{zl} = A_s = A_v = \mathbf{0}_{3\times 1},$$

$$C_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ (1-\gamma) & (\theta-1) & 1 & 0 \\ (1+\omega) & \theta(1+\omega) & 0 & 1 \end{bmatrix}, \quad \text{and} \quad C_s = C_v = \mathbf{0}_{3\times 1}.$$

Comparative Statics and Impulse Responses

Table 6 contains the parameter values used for the baseline calibration. The preference, production, and policy rule parameter values are standard in the literature. We set $q = 1$ to understand the effect of time-varying volatility on the dynamics of inflation and the output gap. We set a persistent process for the volatility shock, with autocorrelation coefficient $\rho_v = 0.5$, and a large ratio for the volatility of the process with respect to its mean, such that $\delta_v = 0.95^{-2}$. Figure 2 presents comparative statics of the factor loadings of the output gap and inflation on monetary policy shocks and stochastic volatility shocks. The comparative statics are presented for the two approximation methods: the traditional linearization method and the log-linearization. The loadings of the output gap and inflation on policy shocks are not significantly different under the two methods, except for the loading of inflation for low reactions in the policy rule to inflation. The loadings on volatility are different. Using the traditional linearization method, the macro variables are not affected by changes in the volatility of the policy shock. Using the log-linearization method, the loadings of the output gap and inflation on volatility imply lower output and higher inflation for an increase in volatility. A higher elasticity of substitution $(1/\gamma)$, a higher Frisch elasticity of labor supply $(1/\omega)$, or an increased persistence in the volatility of policy shocks (ρ_v) increase the sensitivity of the output gap and inflation to volatility. The sensitivity does not seem to be very affected by the reaction to inflation in the policy rule $(\imath_\pi)$.

Figure 3 shows impulse responses of the output gap and inflation to one-standard deviation shocks to u_t and v_t. The response to policy shocks is not affected by the solution method. The log-linearization method shows negative and positive responses of the output gap and inflation, respectively, to a positive volatility shock. The effect of the shock lasts for around 7 periods.

Solution Method Accuracy

We analyze the accuracy of the solution method with and without time-varying volatility using descriptive statistics and the Den Haan-Marcet test. Consider first the errors associated to equations (25), (27), and (28). These errors can be written as

$$\begin{aligned} e_{\pi,t} &= \left(\frac{1+\theta\omega}{1-\theta}\right)\log\left[\frac{1}{1-\alpha}\left(1-\alpha e^{-(1-\theta)\pi_t}\right)\right] + h_t - (\omega+\gamma)x_t - g_t, \\ e_{h,t} &= h_t - \log\left(1+\alpha\beta\mathbb{E}_t[e^{(1-\gamma)\Delta x_{t+1}+(\theta-1)\pi_{t+1}+h_{t+1}}]\right), \\ \text{and} \quad e_{g,t} &= g_t - \log\left(1+\alpha\beta\mathbb{E}_t[e^{(1+\omega)\Delta x_{t+1}+\theta(1+\omega)\pi_{t+1}+g_{t+1}}]\right). \end{aligned} \tag{31}$$

These errors have a closed-from solution given the linear solution for the endogenous variables. We simulate 10,000 observations for stochastic volatility and policy shocks, and solve the model using the linearization and log-linearization methods. We compute the mean and standard deviation of the approximation errors above implied by both methods. Table 7 presents the statistics for models with and without stochastic volatility under the two solution methods. Two points are worth mention. First, the log-linearization method reduces considerably the mean and standard deviation of errors $e_{h,t}$ and $e_{g,t}$ with respect to the traditional linearization method. Second, time variation in volatility does not affect the mean and standard deviation of the approximation error. In fact, the table shows smaller and less volatile errors in the model with time-varying volatility.

Consider now the results of the Den Haan and Marcet (1994) accuracy test. We test the hypothesis that the errors implied by the four equations (13), (18), and (22) are zero. We use the policy shocks u_t as the instrument for the test. The test involves then a chi-square distribution with $4 \times 1 = 4$ degrees of freedom.

Table 8 presents the results for models with and without time-varying volatility using the linearization or log-linearization solution methods. It is immediately clear from the table that with or without stochastic volatility in the model, the log-linearization method is much more accurate than the traditional linearization solution. Under the benchmark parameter values with time-varying volatility, the rejection rates are 0.2% and 0.3% in the lower and upper tails, respectively, using log-linearization, compared to 21.1% and 26.5% in the lower and upper tails, respectively, using the linearization method. The rejection rates are comparable for a model with constant volatility

Panels A, B, an C in Table 8 show the rejection rates for different values for γ, α, and ϕ_u, respectively. The accuracy of the log-linearization solution is highly sensitive to the price rigidity parameter and the autocorrelation coefficient of the shock. In particular, a high price rigidity or highly autocorrelated shock decrease the accuracy of the solution. However, it can be seen that this fact does not depend on whether or not the model has time-varying volatility. Therefore, time variation in volatility of policy shocks does not seem to have a significant effect on the accuracy of the solution method.

4 Conclusion

Time variation in volatility is potentially an important channel to understand empirical regularities in macroeconomic and asset pricing dynamics. This paper shows that the autoregressive gamma process represents an appropriate and highly tractable way to incorporate time-varying volatility to dynamic equilibrium models. A general method to solve these models is provided. The method is applied to the analysis of the Bansal and Yaron asset pricing model and a simple New Keynesian model. The analysis shows that the method is simple, fast, and its accuracy is not negatively affected by the presence of time-varying volatility. Given its generality and tractability, we believe that this method can become an important tool to incorporate time-varying volatility to equilibrium models and analyze its implications on economic dynamics.

A Risk Premia Under Normal and Autoregressive Gamma Processes

Let $m_{t+1} = m_{s,t+1} + m_{v,t+1}$ be the log-pricing kernel, where

$$m_{s,t+1} = \mathbb{E}_t[m_{s,t+1}] - \lambda_s^\top \Sigma(\mathbf{v}_t)^{1/2}\varepsilon_{t+1},$$

and

$$m_{v,t+1} = \mathbb{E}_t[m_{v,t+1}] - \lambda_v^\top(\mathbf{v}_{t+1} - \mathbb{E}_t[\mathbf{v}_{t+1}]),$$

are the conditional normal and autoregressive gamma process components, respectively. The prices of Gaussian risk are contained in vector λ_s and the prices of autoregressive gamma risk are contained in vector λ_v. Similarly, let $r_{t+1} = r_{s,t+1} + r_{v,t+1}$ be the continuously compounded asset return where

$$r_{s,t+1} = \mathbb{E}_t[r_{s,t+1}] + r_s^\top \Sigma(\mathbf{v}_t)^{1/2}\varepsilon_{t+1},$$

and

$$r_{v,t+1} = \mathbb{E}_t[r_{v,t+1}] + r_v^\top(\mathbf{v}_{t+1} - \mathbb{E}_t[\mathbf{v}_{t+1}]),$$

are the conditional normal and autoregressive components, respectively, where r_s and r_v are the vectors of return sensitivities to Gaussian and autoregressive gamma risks, respectively.

From the pricing equation $1 = \mathbb{E}_t[\exp(m_{t+1} + r_{t+1})]$, we can characterize the risk-free rate $R_{f,t}$ by

$$\begin{aligned} \frac{1}{R_{f,t}} = \mathbb{E}_t[M_{t,t+1}] &= \exp\left(\mathbb{E}_t[m_{t+1}] + \frac{1}{2}\mathrm{var}_t(m_{s,t+1})\right) \\ &\times \exp\left(\sum_{i=1}^{N_v}\delta_i\left(\lambda_{z,i}\varsigma_i - \log[1+\lambda_{z,i}\varsigma_i]\right) + \sum_{i=1}^{N_v}\frac{\lambda_{z,i}^2\rho_i\varsigma_i}{1+\lambda_{z,i}\varsigma_i}v_{i,t}\right). \end{aligned} \tag{32}$$

The asset return $R_t = \exp(r_t)$ has expected return

$$\begin{aligned} \mathbb{E}_t[R_{t+1}] &= \exp\left(\mathbb{E}_t[r_{t+1}] + \frac{1}{2}\mathrm{var}_t(r_{s,t+1})\right) \\ &\times \exp\left(-\sum_{i=1}^{N_v}\delta_i\left(r_{z,i}\varsigma_i + \log[1-r_{z,i}\varsigma_i]\right) - \sum_{i=1}^{N_v}\frac{r_{z,i}^2\rho_i\varsigma_i}{1-r_{z,i}\varsigma_i}v_{i,t}\right). \end{aligned} \tag{33}$$

The asset return also satisfies the equation

$$\begin{aligned}\mathbb{E}_t[M_{t,t+1}R_{t+1}] &= \exp\left(\mathbb{E}[m_{t+1}+r_{t+1}]+\frac{1}{2}\mathrm{var}_t(m_{s,t+1}+r_{s,t+1})\right)\\ &\times \exp\left(\sum_{i=1}^{N_v}\delta_i\left((\lambda_{v,i}-r_{v,i})\varsigma_i-\log[1+(\lambda_{v,i}-r_{z,i})\varsigma_i]\right)\right)\\ &\times \exp\left(\sum_{i=1}^{N_v}\frac{(\lambda_{v,i}-r_{v,i})^2\rho_i\varsigma_i}{1+(\lambda_{v,i}-r_{v,i})\varsigma_i}v_{i,t}\right).\end{aligned}\tag{34}$$

Reorganizing terms and replacing equations (32) and (33) in equation (34), the asset return premium is

$$\begin{aligned}\log\left(\frac{\mathbb{E}_t[R_{t+1}]}{R_{f,t}}\right) &= -\mathrm{cov}_t(m_{s,t+1},r_{s,t+1})\\ &+ \sum_{i=1}^{N_v}\delta_i\log\left[\frac{1+(\lambda_{v,i}-r_{v,i})\varsigma_i}{(1+\lambda_{v,i}\varsigma_i)(1-r_{v,i}\varsigma_i)}\right]\\ &- \sum_{i=1}^{N_v}\rho_i\varsigma_i\left(\frac{(\lambda_{v,i}-r_{v,i})^2}{1+(\lambda_{v,i}-r_{v,i})\varsigma_i}-\frac{\lambda_{v,i}^2}{1+\lambda_{v,i}\varsigma_i}+\frac{r_{v,i}^2}{1-r_{v,i}\varsigma_i}\right)v_{i,t}.\end{aligned}\tag{35}$$

Notice that the risk premium can also be written in terms of the continuously compounded excess returns as

$$\log\left(\frac{\mathbb{E}_t[R_{t+1}]}{R_{f,t}}\right)=\mathbb{E}_t[r_{t+1}]-r_t+J.I._t,$$

where, from equation (33), the Jensen's inequality term is

$$J.I._t=\frac{1}{2}\mathrm{var}_t(r_{s,t+1})-\sum_{i=1}^{N_v}\delta_i\left(r_{z,i}\varsigma_i+\log[1-r_{z,i}\varsigma_i]\right)-\sum_{i=1}^{N_v}\frac{r_{z,i}^2\rho_i\varsigma_i}{1-r_{z,i}\varsigma_i}v_{i,t}.$$

B New Keynesian Model - Firm's Problem

Under monopolistic competition and Calvo (1983) staggered price setting, a firm can choose the optimal P_t^* with probability α each period. The firm's optimization problem is

$$\max_{P_t^*} \quad E_t\left[\sum_{s=0}^{\infty} \alpha^s M_{t,t+s}\left(P_t^* Y_{t+s|t}(j) - W_{t+s|t}(j) N_{t+s|t}(j)\right)\right] \tag{36}$$

$$\text{s.t.} \quad Y_{t+s|t}(j) = A_{t+s} N_{t+s|t}(j) \tag{37}$$

$$Y_{t+s|t}(j) = \left(\frac{P_t^*}{P_{t+s}}\right)^{-\theta} Y_{t+s} \tag{38}$$

$$P_t = \left[\int_0^1 P_t(j)^{1-\theta} dj\right]^{\frac{1}{1-\theta}} = \left[(1-\alpha)(P_t^*)^{(}1-\theta) + \alpha P_{t-1}^{(}1-\theta)\right]^{\frac{1}{1-\theta}}. \tag{39}$$

where M is the pricing kernel, Y is output, W is the nominal wage, and N is labor demand. The subscript $t+s|t$ denotes the value in period $t+s$ given that the last price adjustment was in period t. The problem constraints are the production function, where A is labor productivity, and the product demand curve. The first order condition for the firm is:

$$E_t\left[\sum_{s=0}^{\infty} \alpha^s M_{t,t+s}(P_t^* Y_{t+s|t} - \mu S_{t+s|t})\right] = 0, \tag{40}$$

where $\mu = \frac{\theta}{\theta-1}$ is the frictionless markup in the absence of price rigidities, and $S_{t+s|t}$ is the marginal (nominal) cost

$$S_{t+s|t} = \frac{Y_{t+s|t}^{1+\omega}}{A_{t+s}^{1+\omega}} Y_{t+s}^{\gamma} P_{t+s}. \tag{41}$$

From (38) and (41), (40) can be written as

$$E_t\left[\sum_{s=0}^{\infty} \alpha^s M_{t,t+s} Y_{t+s}\left(\frac{P_t^*}{P_{t+s}}\right)^{-\theta} P_t^*\right] = E_t\left[\sum_{s=0}^{\infty} \alpha^s M_{t,t+s}\mu\left(\frac{Y_{t+s}^{1+\omega+\gamma}}{A_{t+s}^{1+\omega}}\right)\left(\frac{P_t^*}{P_{t+s}}\right)^{-\theta(1+\omega)} P_{t+s}\right].$$

The left-hand side of the equation can be written as

$$P_t^*\left(\frac{P_t^*}{P_t}\right)^{-\theta} Y_t H_t, \quad \text{where} \quad H_t = E_t\left[\sum_{s=0}^{\infty} \alpha^s M_{t,t+s}\frac{Y_{t+s}}{Y_t}\left(\frac{P_t}{P_{t+s}}\right)^{-\theta}\right],$$

and the right-hand side of the equation can be written as

$$\mu Y_t^{1+\omega+\gamma}\left(\frac{P_t^*}{P_t}\right)^{-\theta\omega}\left(\frac{P_t}{A_t^{1+\omega}}\right) G_t,$$

where

$$G_t = E_t \left[\sum_{s=0}^{\infty} \alpha^s M_{t,t+s} \left(\frac{A_t}{A_{t+s}} \right)^{1+\omega} \left(\frac{Y_{t+s}}{Y_t} \right)^{1+\omega+\gamma} \left(\frac{P_t}{P_{t+s}} \right)^{-\theta\omega-1} \right].$$

H_t and G_t can be written recursively as in equations (27) and (28), respectively. It follows that the optimality condition can be written as

$$\left(\frac{P_t^*}{P_t} \right)^{1+\theta\omega} H_t = X_t^{\omega+\gamma} G_t.$$

Finally, from the identity

$$\frac{P_t^*}{P_t} = \left[\frac{1}{1-\alpha} \left(1 - \alpha \Pi_t^{-(1-\theta)} \right) \right]^{\frac{1}{1-\theta}},$$

we obtain (25).

C Den Haan-Marcet Accuracy Test

To test the accuracy of the log-linear approximation of the rational expectation models, we employ the Den Haan and Marcet (1994) test via simulations. Den Haan-Marcet test exploits the fact that for a system of expectational equations

$$f(z_t) = E(\phi(z_{t+1}, z_{t+2}, \ldots)|\Omega_t),$$

where z_t is a vector of endogenous and exogenous variables, and $f : \mathbb{R}^n \to \mathbb{R}^m$ and $\phi : \mathbb{R}^n \times \mathbb{R}^\infty \to \mathbb{R}^m$ are well defined functions. The residuals, defined as $u_{t+1} = \phi(z_{t+1}, z_{t+2}, \ldots) - f(z_t)$, satisfy

$$E[u_{t+1} \otimes h(x_t)] = 0,$$

for any k-dimensional, $t-$measurable vector x_t and function $h : R^k \to R^q$. The Den Haan-Marcet test produces the statistics

$$T B_T^\top A_T^{-1} B_T \to \chi^2(qm) \quad \text{as} \quad T \to \infty$$

using a large number of realizations of simulated data, $\bar{u}_t$ and $\bar{x}_t$ where

$$B_T \equiv \frac{\sum_T \bar{u}_{t+1} \otimes h(\bar{x}_t)}{T},$$

and A_T is a consistent estimator of

$$S = \sum_{-\infty}^{\infty} E[[u_{t+1} \otimes h(x_t)]\dot{E}[u_{t+1} \otimes h(x_t)]^{\top}].$$

If the resulting test statistics belongs to either the upper or lower critical tails of the χ^2 distribution, then we can reject the hypothesis that the approximate solution is accurate. We can repeat the test a large number of times to obtain the percentage of rejections for a given model. For both the Bansal and Yaron model and the New-Keynesian model presented above, we choose to simulate the solution for 1200 periods (100 years) and repeat the simulation 1000 times each to construct the rejection probabilities in tables 5 and 8.

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Table 1: **Bansal and Yaron Model - Baseline Parameter Values**

Parameter	Description	Value
β	Subjective discount factor	0.999
ψ	Elasticity of intertemporal substitution	1.5
γ	Coefficient of relative risk aversion	10
μ_c	Average consumption growth	0.0015
σ_c	Volatility parameter for consumption growth	0.0078
ϕ_x	Autocorrelation parameter for x_t	0.979
$\sigma_x \times 10^4$	Volatility parameter for x_t	3.4320
μ_d	Average dividend growth	0.0015
ϕ_{dx}	Loading of dividend growth on x_t	3
σ_d	Volatility parameter for dividend growth	0.0351
$c_v \times 10^4$	Parameter of time-varying volatility	7.1925
ρ_v	Autocorrelation parameter of time-varying volatility	0.987
δ_v	Parameter of time-varying volatility	18.07

Table 2: **Bansal and Yaron Model - Annualized Time Average Growth Rates**

The model parameter values are presented in Table 1. The data statistics are obtained from Bansal and Yaron (2004). The model statistics are mean values for 3,000 simulations each with 840 monthly observations that are aggregated to an annual frequency. $AC(u,i)$ denotes the i-th autocorrelation for variable u, and $corr$ denotes correlations.

Variable	Data	Model
$\sigma(\Delta c)$	2.93	2.81
$AC(\Delta c, 1)$	0.49	0.46
$AC(\Delta c, 2)$	0.15	0.22
$\sigma(\Delta d)$	11.49	11.16
$AC(\Delta d, 1)$	0.21	0.36
$corr(\Delta c, \Delta d)$	0.55	0.29

Table 3: **Bansal and Yaron Model - Asset Pricing Implications**

The model baseline parameter values are presented in Table 1. The data statistics are obtained from Bansal and Yaron (2004). The model statistics are mean values for 3,000 simulations each with 840 monthly observations that are aggregated to an annual frequency. The expressions $\mathbb{E}[r_m - r]$, $\mathbb{E}[r]$, and $\mathbb{E}[p]$ are, respectively, the annualized equity premium, average risk-free rate and price-dividend ratio. The annual price-dividend ratio was constructed taking the price for the last month of the year and accumulating monthly dividends to be paid at the end of the year (assuming a reinvestment rate of 0).

Variable	Data	Model	
		$\gamma = 7.5$	$\gamma = 10$
$\mathbb{E}[r_m - r]$	6.33	4.52	6.02
$\sigma(r_m - r)$	19.42	16.99	16.67
$\mathbb{E}[r]$	0.86	1.62	1.34
$\sigma(r)$	0.97	1.2	1.20
$\sigma(p)$	0.29	0.19	0.19
$AC(p, 1)$	0.81	0.66	0.65

Table 4: **Bansal and Yaron Model - Solution Method Error Analysis**

This table reports the average and standard deviation of the error equations (23) for 10,000 simulations of the exogenous variables. The model baseline parameter values are presented in Table 1. The statistics are computed for models with constant volatility ($I_v = 0$) and time-varying volatility ($I_v = 1$).

	Constant volatility		Time-varying volatility	
Error	Mean	Std. Dev.	Mean	Std. Dev.
$e_{w,t}$	3.28×10^{-7}	4.54×10^{-7}	3.93×10^{-7}	5.35×10^{-7}
$e_{p,t}$	3.14×10^{-5}	1.42×10^{-3}	4.77×10^{-5}	1.46×10^{-3}

Table 5: **Bansal and Yaron Model - Accuracy of the Solution Method**

All reported numbers are the percentage of the Den Haan-Marcet (1994) test statistics in 1,000 simulations of 1,200-observation time series that are below the 5th percentile or above the 95th percentile of the $\chi^2(6)$ distribution. Panel A shows the rejection rate by varying the coefficient of risk aversion while keeping all other parameters at their benchmark values. Panel B shows the rejection rate by varying the elasticity of intertemporal substitution. Panel C shows the rejection rate by varying the autoregressive coefficient on long-run risk. The numbers reported with and without parentheses refer to the rejection rates for models with time-varying and constant volatility, respectively.

Panel A $\psi = 1.5$, $\phi_x = 0.979$	$\gamma = 7.5$	$\gamma = 10$	$\gamma = 12.5$
lower 5%	27.1	8.4	1.1
	(21.9)	(11.6)	(2.4)
upper 5%	0	0.3	5.4
	(0)	(0.6)	(3.6)
Panel B $\gamma = 10$, $\phi_x = 0.979$	$\psi = 1.5$	$\psi = 0.99$	$\psi = 0.67$
lower 5%	8.3	30.1	14.1
	(11.4)	(33.1)	(18.9)
upper 5%	0.4	2.2	14.3
	(0.4)	(1.8)	(14.1)
Panel C $\gamma = 10$, $\psi = 1.5$	$\phi_x = 0.959$	$\phi_x = 0.969$	$\phi_x = 0.979$
lower 5%	8.6	10.5	8.4
	(12.4)	(10.9)	(11.6)
upper 5%	0	0.3	0.3
	(0.2)	(0)	(0.6)

Table 6: **New Keynesian Model - Baseline Parameter Values**

Parameter	Description	Value
β	Subjective discount factor	0.99
γ	Coefficient of relative risk aversion	2.5
ω	Inverse of Frisch labor elasticity	0.8
α	Degree of price rigidity	0.66
θ	Elasticity of substitution of goods	10
ϕ_u	Autocorrelation of policy shocks	0.7
σ_u	Volatility parameter for policy shocks	0.002
c_v	Volatility parameter for policy shocks	0.45125
ρ_v	Volatility parameter for policy shocks	0.5
δ_v	Volatility parameter for policy shocks	0.95^{-2}
$\bar{\imath}$	Constant in the policy rule	0.01
$\imath_\pi$	Response to inflation in the policy rule	1.5
q	Sensitivity of policy shocks to time-varying volatility	1

Table 7: **New Keynesian Model - Solution Method Error Analysis**

	Constant volatility		Time-varying volatility	
Error	Mean	Std. Dev.	Mean	Std. Dev.
Linearization method				
$e_{\pi,t}$	6.33×10^{-3}	1.45×10^{-2}	4.48×10^{-3}	1.43×10^{-2}
$e_{h,t}$	5.04×10^{-1}	1.67×10^{-2}	5.01×10^{-1}	1.61×10^{-2}
$e_{g,t}$	5.05×10^{-1}	2.01×10^{-2}	5.03×10^{-1}	1.94×10^{-2}
Log-linearization method				
$e_{\pi,t}$	6.39×10^{-3}	1.02×10^{-2}	4.96×10^{-3}	8.87×10^{-3}
$e_{h,t}$	7.7×10^{-4}	1.16×10^{-3}	5.93×10^{-4}	9.99×10^{-4}
$e_{g,t}$	1.16×10^{-3}	1.76×10^{-3}	9.52×10^{-4}	1.61×10^{-3}

Table 8: **New Keynesian Model - Accuracy of the Solution Method**

Reported numbers are the percentage of the Den Haan-Mercet (1994) test statistics in 1,000 simulations of 1,200-observation time series that are below the 5th percentile or above the 95th percentile of the $\chi^2(4)$ distribution. Panels A, B, and C shows the rejection rate by varying, respectively, the coefficient of risk aversion, the degree of price rigidity, the autoregressive coefficient of monetary policy shocks, while keeping all other parameters at their benchmark values in Table 6. Numbers reported with and without parentheses refer to the rejection rates for models with time-varying and constant volatility, respectively.

Panel A				
$\alpha = 0.66$, $\phi_u = 0.7$		$\gamma = 1$	$\gamma = 2.5$	$\gamma = 5$
Linearization	lower 5%	14.5	21.1	22.9
		(15.3)	(22.0)	(21.7)
	upper 5%	26.9	26.5	26.7
		(32.4)	(28.4)	(26.3)
Log-linearization	lower 5%	1.1	0.2	0.2
		(0)	(0)	(0)
	upper 5%	0.8	0.3	0.8
		(0.8)	(1.0)	(0.6)
Panel B				
$\gamma = 2.5$, $\phi_u = 0.7$		$\alpha = 0.5$	$\alpha = 0.66$	$\alpha = 0.8$
Linearization	lower 5%	0	21.1	16.6
		(0)	(22.0)	(16.2)
	upper 5%	62.1	26.5	20.0
		(63.7)	(28.4)	(21.0)
Log-linearization	lower 5%	0.4	0.2	29.7
		(0)	(0)	(34.7)
	upper 5%	4.1	0.3	0.4
		(6.8)	(1.0)	(0)
Panel C				
$\gamma = 2.5$, $\alpha = 0.66$		$\phi_u = 0.5$	$\phi_u = 0.7$	$\phi_u = 0.9$
Linearization	lower 5%	33.2	21.1	19.7
		(30.9)	(22.0)	(23.1)
	upper 5%	6.7	26.5	1.2
		(8.4)	(28.4)	(0.9)
Log-linearization	lower 5%	49.0	0.2	0.1
		(44.4)	(0)	(0)
	upper 5%	0.4	0.3	57.4
		(0.2)	(1.0)	(71.6)

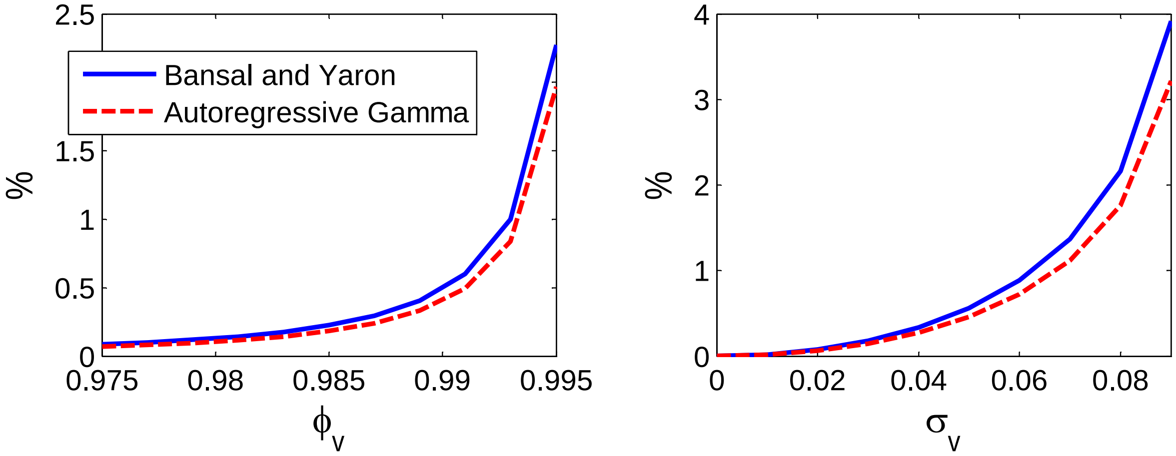

Figure 1: Comparison of the volatility premium under the approximate square root and autoregressive gamma specifications for volatility for the Bansal and Yaron model.

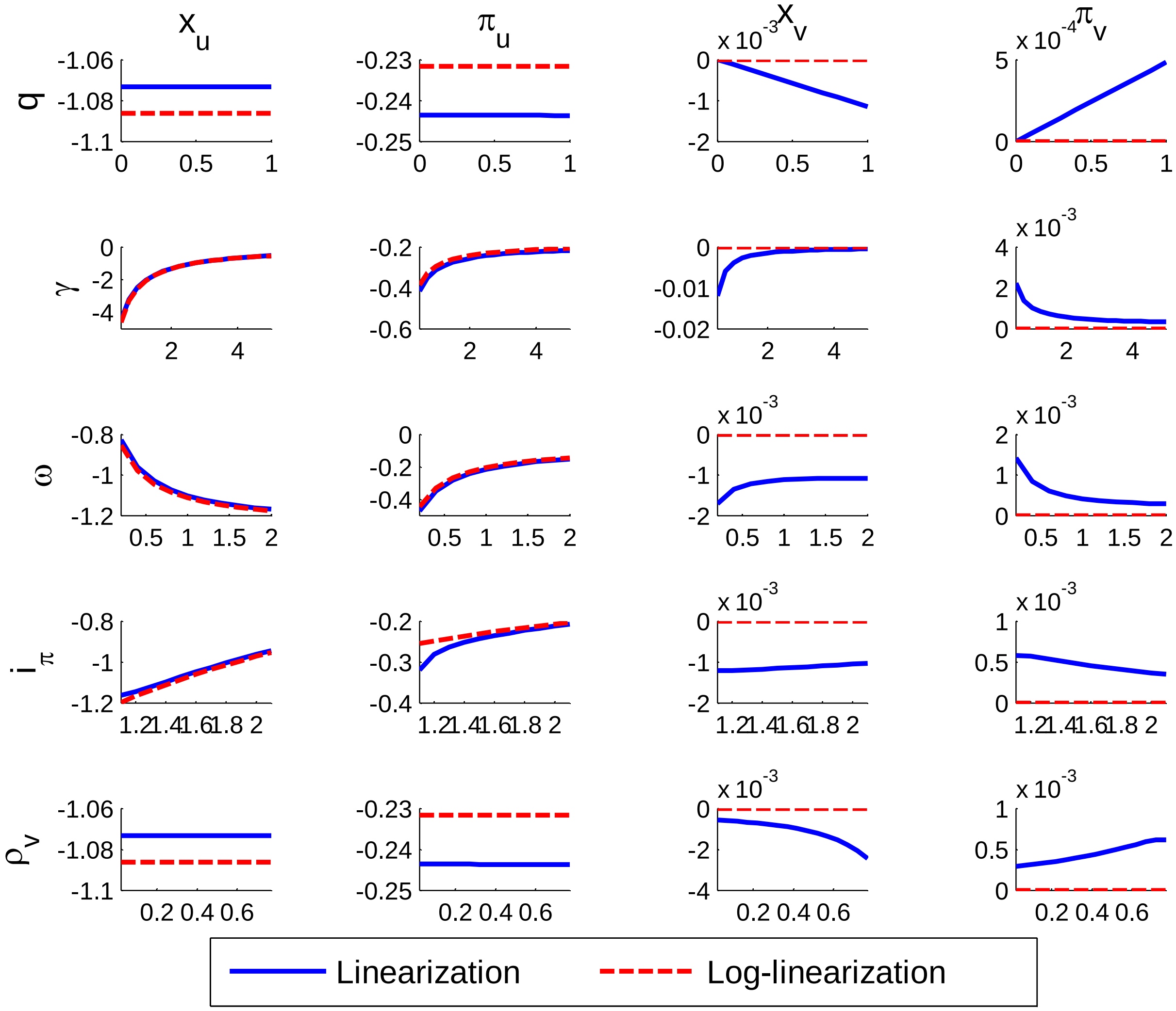

Figure 2: Comparative statics of loadings for the output gap and inflation on the state variables changing q, γ, ω, $\imath_\pi$, and ρ_v. The baseline parameter values are presented in Table 6.

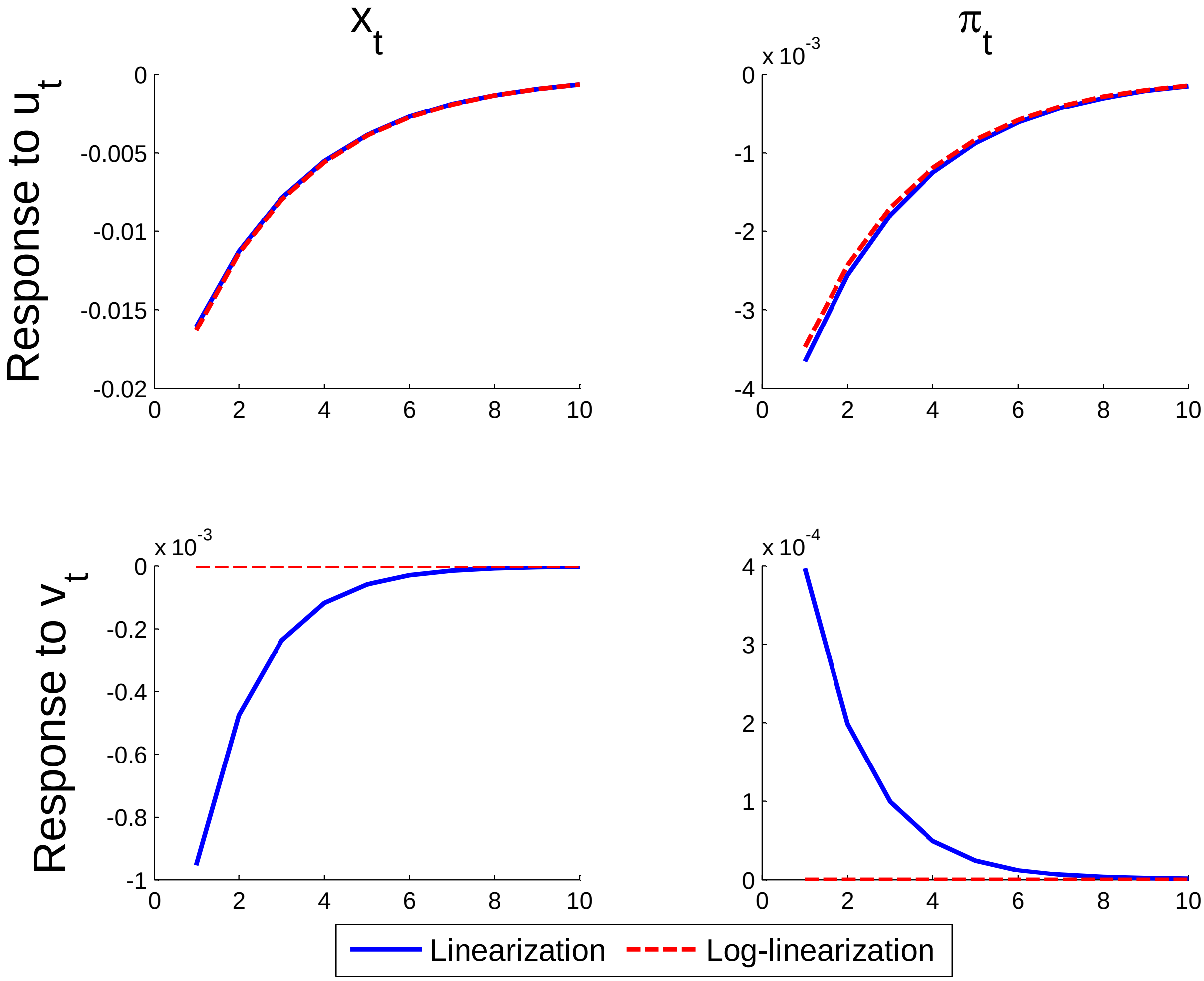

Figure 3: Impulse responses for the output gap and inflation to a one-standard deviation positive policy shock. The baseline parameter values are presented in Table 6.