Implications of Monetary Policy on the Real Yield Curve and the Inflation Risk Premia

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Preliminary. Comments Welcome!

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Abstract

The importance in finance and economics of real interest rates of short and long maturities contrasts with our limited understanding of their economic foundations. We provide a theoretical analysis of the implications of monetary policy on the real term structure of interest rates and the inflation risk premia in nominal bonds. Monetary policy has real effects in an economy characterized by recursive preferences, product price rigidities, and a nominal interest-rate policy rule. Productivity growth shocks and policy shocks generate a positive covariance between output and inflation, affect the marginal utility of consumption in real and nominal terms, and thus the joint dynamics of real and nominal bonds. Welfare-improving policies imply more negative term premia, less negative inflation risk premia, and a higher correlation between real returns on real and nominal bonds. As a result, more responsive monetary policies reduce the government borrowing savings of nominal vs. real debt, and reduce the diversification benefits of real bonds.

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1 Introduction

The concept of the risk-free rate plays a fundamental role in asset pricing, corporate finance, and macroeconomics. This rate contains significant information about the stochastic discount factor, represents a benchmark return to understand the valuation of risks in financial assets and the cost of capital of investment projects, and provides a measure of the willingness to substitute consumption over time in the economy. A truly risk-free rate is riskless in real terms rather than in terms of money since the value of consumption in terms of money may vary over time. Nominal rates then may command a compensation for changes in this value, or an inflation risk premium. In spite of their importance, our understanding of the economic foundations of the risk-free rate and the inflation risk premium is very limited. This paper provides a theoretical analysis of the term structure of real risk-free rates and the inflation risk premia in nominal bonds, their link to monetary policy and welfare. The analysis is used to provide insights into the volatility of long-term real yields, the government borrowing costs, and the diversification benefits of real bonds.

Monetary policy is a natural element to incorporate to the joint study of the real yield curve and inflation risk premia. The two most common objectives of monetary policy are to promote conditions for price stability and maximum employment. The policy then becomes a determinant of inflation and economic activity, whose effects may be reflected on real and nominal interest rates of short and long maturities. Since the inflation risk premium is a compensation for risks that simultaneously affect real and nominal variables, monetary policy is potentially an important driver of this premium.

We model a production economy characterized by a representative agent with recursive preferences, a production sector with nominal price rigidities, a monetary policy rule, and two sources of uncertainty. Recursive preferences allow us to break the strong link between risk aversion and intertemporal substitution implied by preferences with constant relative risk aversion. It gives us additional flexibility to simultaneously capture asset pricing dynamics and macroeconomic behavior, as shown by Tallarini (2000). Nominal price rigidities generate real effects of monetary policy. In the absence of price rigidities, the dynamics of real yields is completely independent from monetary policy. The policy is modeled as a rule to set the short-term nominal rate as a function of current economic conditions. It allows us to analyze the impact of different policy responses to the state of economy on real yields and inflation risk premia. The two sources of uncertainty are shocks to productivity growth and shocks in the monetary policy rule. Equilibrium in the economy implies affine term structures for real and nominal bonds where the driving factors are productivity growth and policy shocks, and the loadings on these factors depend on preference, production, and policy parameters.

Monetary policy in the model economy has effects on short- and long-term real bonds, the inflation risk premium, the volatility of long-term yields, and the correlation between real and nominal bonds. As a result of distortions in production generated by price rigidities, the dynamics of the short-term real rate is affected by productivity growth and policy shocks. Negative shocks to productivity growth or negative policy shocks decrease the level of the short-term rate. These shocks reduce consumption and imply lower real yields for long-term bonds. That is, the price of long-term real bonds and the marginal utility of consumption are simultaneously high. Real bonds then involve a negative term premium since they provide a hedge for consumption risk. The term premium is sensitive to the response to economic conditions in the policy rule. A stronger response to output and inflation imply lower distortions in consumption, which translates into more negative term premia. The inflation risk premium in the model is also negative. This is the result of a positive covariance between inflation and output induced by shocks to productivity growth and policy shocks. This covariance declines as the policy rule becomes more sensitive to economic conditions, implying a less negative inflation risk premium. The volatility of real bond yields is also affected by the policy. Policy rules with a weak response to inflation and output imply a significant volatility in inflation and consumption which in turn implies high volatilities for real and nominal yields. When the responses in the policy rule increase, distortions in output and inflation are smaller and the volatilities of real and nominal rates decline and tend to converge.

Campbell and Shiller (1996) propose evaluating the convenience for the economy of indexed bonds in several dimensions. We use our economic model to link three of these dimensions to monetary policy and welfare. First, in the absence of real term premia (relatively constant real rates), a short-term nominal bond involves investment opportunities similar to those of long-term bonds. However, the negative term premium in the model imply that long-term bonds provide advantages over short-term nominal bonds since they allow investor to hedge the risk of changes in real rates. The term premia become more negative for stronger responses of monetary policy to economic conditions, which in turn increase welfare. Therefore, the benefits of long-term bonds are higher in economies with higher welfare. Second, the inflation risk premia can be seen as the difference in real borrowing costs for the government of issuing nominal bonds over issuing real bonds. Since the two shocks in the model imply a negative inflation risk premium, nominal bonds represent a cheaper financing option for the government over debt in real terms. The savings for the government are higher for policies that involve lower welfare. Third, we provide a partial analysis of the diversification benefits of real bonds considering a constrained investor who has access to a nominal bond and a real bond. The correlation of real returns of nominal and real bonds increase as the response to inflation and output in the policy rule increases. Therefore, access to a real bond is more valuable for the constrained investor in economies with low responses to economic conditions and low welfare.

Related Literature

[TO BE WRITTEN] Campbell and Shiller (1996), Campbell, Shiller and Viceira (2009), Ang, Bekaert and Wei (2008), Rudebusch and Swanson (2009).

The paper is organized as follows. Section 2 presents some descriptive statistics for nominal government bonds and treasury inflation protected securities. The principal component analysis in this section shows that two factors capture most of the variability of nominal and TIPS yields. Section 3 describes the economic model and its equilibrium conditions. Section 4 shows that the real and term structures of the model can be characterized as affine term structures and provide the parameter restrictions imposed by equilibrium. Section 5 presents the analysis and section 6 concludes. The appendix contains all the proofs.

2 Some Descriptive Statistics

The data consist of United States time series for real and nominal bond yields, consumption, and consumer prices from 1999 to 2008. The term structure series was obtained from monthly data on bond yields for yearly maturities from 1 to 20 years. The nominal and real yields are obtained following the procedure in Gurkaynak, Sack and Wright (2006) and Gurkaynak, Sack and Wright (2008), respectively. These data are published on the Federal Reserve website. The short-term nominal interest rate is the 3-month T-bill from the Fama risk-free rates database. The consumption growth series was constructed using quarterly data on real per-capita consumption of nondurables and services from the Bureau of Economic Analysis. The inflation series was constructed following the methodology used in Piazzesi and Schneider (2007) to capture inflation related to non-durables and services consumption only. TIPS with maturities between 2 and 4 years are only available since 2004. Table 1 reports the average yields and the standard deviation of yields for TIPS and nominal bonds. We report statistics computed for the sample $1999-2008$ and $2004-2008$, given the concerns about liquidity in the TIPS market in the early period. The table shows upward sloping average curves and similar volatilities for TIPS and comparable nominal bonds. The nominal yield curve is steeper than the TIPS curve. A positive average slope

Table 1: Average Levels (%), and Standard Deviations (%) for U.S. Government TIPS and Nominal Bond Yields

	1999 - 2008		2004 - 2008	
	Real	Nominal	Real	Nominal
Average Yields				
2 years		3.79	1.22	3.64
5 years	2.30	4.32	1.60	3.98
10 years	2.66	4.97	2.00	4.54
15 years	2.78	5.31	2.15	4.86
20 years	2.81	5.41	2.17	4.95
Standard Deviations				
2 years		1.50	1.06	1.09
5 years	1.11	1.05	0.63	0.65
10 years	0.87	0.73	0.34	0.33
15 years	0.77	0.62	0.25	0.29
20 years	0.75	0.58	0.22	0.32

for real bonds represents a significant challenge for consumption-based asset pricing models.

Table 2 shows the variability of nominal bond and TIPS yields captured by their three principal components when these components are computed for TIPS and nominal bonds separately and jointly. The fact that the two principal components in the joint analysis can capture most of the variability of TIPS and nominal yields suggest a two-factor model to understand the joint dynamics of nominal and real bonds. Section 3 presents an economic model that implies the two-factor equilibrium affine term structure model for nominal and real yields presented in section 4.

3 Economic Model

We model a production economy where households derive utility from the consumption of a basket of goods and disutility from supplying labor for production. The production sector is characterized by monopolistic competition and nominal price rigidities. These rigidities generate real effects of monetary policy. When some producers are not able to adjust prices, inflation generates distortions that affect production decisions. Since inflation is determined by monetary policy, different policies have different implications for real activity, and real and nominal interest rates for all maturities.

Table 2: Variability (%) explained by the three principal components for U.S. Government TIPS and Nominal Bond Yields

	1999 - 2008			2004 - 2008		
	Real	Nominal	All	Real	Nominal	All
1st	98.04	78.46	81.88	90.41	70.63	75.63
2nd	1.92	20.89	15.50	9.09	28.33	21.2
3rd	0.03	0.44	2.05	0.44	0.53	2.19
Total	99.99	99.79	99.43	99.94	99.49	99.02

We model monetary policy as an interest-rate policy rule that reacts to economic conditions.[1]

3.1 Household

The representative agent in this economy chooses consumption C_t and labor N_t to maximize the Epstein and Zin (1989) recursive utility function

$$V_t = \left\{ (1-\beta)U(C_t, N_t)^{1-\psi} + \beta \mathbb{E}_t \left[V_{t+1}^{1-\gamma} \right]^{\frac{1-\psi}{1-\gamma}} \right\}^{\frac{1}{1-\psi}}, \tag{1}$$

where $0 < \beta < 1$ is the subjective discount factor, ψ^{-1} is the elasticity of intertemporal substitution, γ is the coefficient of relative risk aversion. The recursive utility formulation allows us to relax the strong condition $\gamma = \psi$ implied by constant relative risk aversion. The intratemporal utility of consumption, C_t, and aggregate labor, N_t, is

$$U(C_t, N_t) = \left(\frac{C_t^{1-\psi}}{1-\psi} - \frac{N_t^{1+\omega}}{1+\omega} \right)^{\frac{1}{1-\psi}},$$

where ω^{-1} is the elasticity of substitution of labor. The consumption good is a basket of differentiated goods produced in a continuum of industries. Specifically, consumption of the final good is

$$C_t = \left[\int_0^1 C_t(j)^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}},$$

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[1] The model can be seen as an extension of the standard New-Keynesian framework (see Woodford (2003), for instance) that incorporates recursive preferences for households.

where $\theta > 1$ is the elasticity of substitution across differentiated goods, and $C_t(j)$ is the consumption of the differentiated good j. Labor is the aggregate of industry labors given by

$$N_t = \left[\int_0^1 N_t(j)^{1+\omega} dj\right]^{\frac{1}{1+\omega}} .$$

The representative agent is subject to the intertemporal budget constrain

$$\mathbb{E}_t \left[\sum_{s=0}^{\infty} M^{\$}_{t,t+s} P_{t+s} C_{t+s}\right] \leq \mathbb{E}_t \left[\sum_{s=0}^{\infty} M^{\$}_{t,t+s} \left(\int_0^1 W_{t+s}(j) N_{t+s}(j) dj + P_{t+s}\Psi_{t+s}\right)\right],$$

where $M^{\$}_{t,t+s}$ is the nominal discount factor for cashflows at time $t+s$, P_t is the nominal price of a unit of the basket of goods, $W_t(j)$ is the nominal wage earned in the production of good j and Ψ_t is the aggregate profits from production.

The appendix shows that the household's optimality conditions imply the one-period real stochastic discount factors given by

$$M_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\psi} \left(\frac{V_{t+1}}{\mathbb{E}_t[V_{t+1}^{1-\gamma}]^{1/(1-\gamma)}}\right)^{\psi-\gamma}, \tag{2}$$

and the nominal discount factor

$$M^{\$}_{t,t+1} = M_{t,t+1} \left(\frac{P_{t+1}}{P_t}\right)^{-1}. \tag{3}$$

The real and nominal discount factors allow us to price real and nominal default-free bonds in section 4. In particular, a one-period nominal bond has the price

$$e^{-i_t} = \mathbb{E}_t \left[M^{\$}_{t,t+1}\right], \tag{4}$$

where the one-period interest rate i_t is the instrument of monetary policy.

3.2 Firms

The production of differentiated goods is characterized by monopolistic competition and price rigidities. Producers have market power to set the price of their differentiated goods in a Calvo (1983) staggered price setting. That is, a producer is able to change the product price at any period of time, with a probability $1-\alpha$. When the producer is able to adjust the price optimally, the price is set to maximize the present value of profits. The maximization problem can then be written as

$$\max_{\{P_t(j)\}} \mathbb{E}_t \left\{ \sum_{s=0}^{\infty} \alpha^s M^{\$}_{t,t+s} \left[P_t(j) \left(\Pi^*\right)^s Y_{t+s|t}(j) - W_{t+s|t}(j) N_{t+s|t}(j) \right] \right\},$$

subject to the production function

$$Y_{t+s|t}(j) = A_{t+s} N_{t+s|t}(j),$$

and the demand function

$$Y_{t+s|t}(j) = \left(\frac{P_t(j) \left(\Pi^*\right)^s}{P_{t+s}} \right)^{-\theta}.$$

The objective function captures the idea that the producer takes into account the probability of not being able to adjust the price optimally in the future. In such a case the price is adjusted to incorporate the constant target inflation Π^*. The only-labor production function depends on labor productivity, A_t. We model productivity growth, $\Delta a_t \equiv \log A_t - \log A_{t-1}$, as the exogenous process

$$\Delta a_{t+1} = (1-\phi_a)\theta_a + \phi_a \Delta a_t + \sigma_a \varepsilon_{a,t+1},$$

where $\varepsilon_{a,t} \sim \mathrm{IID}\mathcal{N}(0,1)$.

The appendix shows that the solution to this maximization problem involves solving the equation

$$\left[\frac{1}{1-\alpha} \left(1 - \alpha \left(\frac{\Pi_t}{\Pi^*} \right)^{-(1-\theta)} \right) \right]^{\frac{1+\theta\omega}{1-\theta}} H_t = X_t^{\omega+\psi} G_t, \tag{5}$$

where X_t is defined as the deviation of total output, Y_t from the "benchmark" output resulting from an economy under flexible prices, Y_t^f. That is,

$$X_t \equiv \frac{Y_t}{Y_t^f}.$$

The processes H_t and G_t are described by the recursive equations

$$\begin{aligned} H_t &= 1 + \alpha \Pi^* \mathbb{E}_t \left[M^{\$}_{t,t+1} \frac{Y_{t+1}}{Y_t} \left(\frac{\Pi_{t+1}}{\Pi^*} \right)^{\theta} H_{t+1} \right], \\ \text{and} \quad G_t &= 1 + \alpha \Pi^* \mathbb{E}_t \left[M^{\$}_{t,t+1} \frac{Y_{t+1}}{Y_t} \left(\frac{X_{t+1}}{X_t} \right)^{\omega+\psi} \left(\frac{\Pi_{t+1}}{\Pi^*} \right)^{\theta(1+\omega)} G_{t+1} \right], \end{aligned}$$

respectively.

3.3 Monetary Policy

We model a monetary authority that sets the level of a short-term nominal interest rate. Monetary policy is described by the policy rule

$$i_t = \bar{\imath} + \imath_x x_t + \imath_\pi \pi_t + u_t, \tag{6}$$

where the one-period nominal interest rate, i_t, is set responding to aggregate inflation, the output gap, and a policy shock u_t. The coefficients $\imath_x$, and $\imath_\pi$ capture the response of the monetary authority to economic conditions. The policy shock follows the process

$$u_{t+1} = \phi_u u_t + \sigma_u \varepsilon_{u,t+1},$$

where $\varepsilon_{u,t} \sim \mathrm{IID}\mathcal{N}(0,1)$.

3.4 Equilibrium

Equilibrium involves solving for allocations and prices that simultaneously satisfy the system of equations formed by the household's optimality condition (4), the firm's optimality condition (5) and the monetary policy rule (6), subject to the equilibrium constraints that consumption has to be equal to production ($Y_t = C_t$) and labor supply has to be equal to labor demand. The dynamics for equilibrium quantities and prices can be described in terms of productivity growth and policy shocks. In particular, the appendix shows that, up to an approximation, the output gap $x_t \equiv \log X_t$ and inflation $\pi_t \equiv \log P_t - \log P_{t-1}$ follow the processes are presented

$$\begin{aligned} x_t &= \bar{x} + x_a \Delta a_t + x_u u_t, \\ \text{and} \quad \pi_t &= \pi^* + \bar{\pi} + \pi_a \Delta a_t + \pi_u u_t, \end{aligned}$$

where the coefficients are constants that satisfy the equilibrium conditions and depend on deep economic parameters. It follows that output growth $\Delta y_t \equiv \log Y_t^f - \log Y_{t-1}^f$ can be written in terms of the flexible-price output growth $\Delta y_t^f \equiv \log Y_t^f - \log Y_{t-1}^f$ and changes in the output gap, as $\Delta y_t = \Delta y_t^f + \Delta x_t$, where the flexible-price output growth process is

$$\Delta y_{t+1}^f = (1-\phi_a)\theta_y + \phi_a \Delta y_t^f + \sigma_y \varepsilon_{a,t+1},$$

with $\theta_y = \frac{1+\omega}{\omega+\psi}\theta_a$, and $\sigma_y = \frac{1+\omega}{\omega+\psi}\sigma_a$. This process is only affected by productivity shocks, and inherits the persistence of the productivity growth process. In the absence of price rigidities the output gap is zero and production is not affected by policy shocks.

Equilibrium process for output and inflation allow us to obtain the real and nominal stochastic discount factors in equations (2) and (3), respectively, once the process for the continuation utility V_t in equation (1) is determined. This process affects the discount factors when the coefficient of risk aversion γ is different than the inverse of the elasticity of intertemporal substitution ψ^{-1}. The appendix shows that the continuation utility with respect to consumption, $v_t \equiv \log V_t - \log C_t$, can be approximated as

$$\begin{aligned} v_t &= \bar{\eta}_v + \eta_{vx} x_t + \eta_{vv} \log \mathbb{E}_t \left\{ \exp\left[(1-\gamma)(v_{t+1} + \Delta y_{t+1}) \right] \right\} \\ &= \bar{v} + v_a \Delta a_t + v_u u_t. \end{aligned}$$

The approximation parameters $\bar{\eta}_v$, η_{vx}, and η_{vv} are described in the appendix. The constant coefficients $\bar{v}$, v_a, and v_u depend on consumption, production, and policy parameters. Continuation utility depends on policy shocks when prices are not perfectly flexible. Since v_t can be considered as a measure of economic welfare, it allows us to provide a link between welfare, different policy rules and their implied dynamics for asset prices.

4 Term Structures of Real and Nominal Interest Rates

The linear solutions for the output and inflation process allow us to obtain affine representations for the real and nominal discount factors, which in turn imply affine term structures for real and nominal rates. This section shows that equilibrium real and nominal bond yields in the economy above can be expressed as linear functions of productivity growth and policy shocks.

It is convenient to group productivity growth and policy shocks in the state vector $\mathbf{s}_t =$

$(\Delta a_t, u_t)^\top$, which follows the first-order vector autoregression

$$\mathbf{s}_{t+1} = (\mathbb{I} - \Phi)\theta_s + \Phi \mathbf{s}_t + \Sigma^{1/2}\varepsilon_{t+1}, \tag{7}$$

where $\mathbb{I}$ is the 2×2 identity matrix,

$$\Phi = \begin{bmatrix} \phi_a & 0 \\ 0 & \phi_u \end{bmatrix}, \quad \theta_s = \begin{pmatrix} \theta_a \\ 0 \end{pmatrix}, \quad \Sigma^{1/2} = \begin{bmatrix} \sigma_a & 0 \\ 0 & \sigma_u \end{bmatrix}, \quad \text{and} \quad \varepsilon_t = \begin{pmatrix} \varepsilon_{a,t} \\ \varepsilon_{u,t} \end{pmatrix}.$$

The appendix shows that the real stochastic discount factor $M_{t,t+1}$ in equation (2) can be written as

$$-\log M_{t,t+1} = \Gamma_0 + \Gamma^\top \mathbf{s}_t + \lambda^\top \Sigma^{1/2}\varepsilon_{t+1},$$

where $\Gamma = (\Gamma_a, \Gamma_u)^\top$, $\lambda = (\lambda_a, \lambda_u)^\top$,

$$\begin{aligned}
\Gamma_0 &= -\log\beta - \left[\frac{\psi-\gamma}{(1-\gamma)\eta_{vv}}\right]\left[\bar{\eta}_v + \eta_{vx}\bar{x} - (1-\psi)\bar{v}\right] - (\psi-\gamma)\bar{v} \\
&+ \left[\gamma(1+x_a) - (\psi-\gamma)v_a\right](1-\phi_a)\theta_y, \\
\Gamma_a &= -\left[\left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{\eta_{vx}}{\eta_{vv}} + \gamma\right]x_a + \left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{1-\psi}{\eta_{vv}}v_a + \left[\gamma(1+x_a) - (\psi-\gamma)v_a\right]\phi_a, \\
\Gamma_u &= -\left[\left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{\eta_{vx}}{\eta_{vv}} + \gamma\right]x_u + \left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{1-\psi}{\eta_{vv}}v_u + \left[\gamma x_u - (\psi-\gamma)v_u\right]\phi_u, \\
\lambda_a &= \gamma(1+x_a) - (\psi-\gamma)v_a, \\
\text{and} \quad \lambda_u &= \gamma x_u - (\psi-\gamma)v_u.
\end{aligned}$$

Similarly, the nominal discount factor in equation (3) is

$$-\log M^\$_{t,t+1} = \Gamma^\$_0 + \Gamma^{\$\top}\mathbf{s}_t + \lambda^{\$\top}\Sigma^{1/2}\varepsilon_{t+1},$$

where

$$\begin{aligned}
\Gamma^\$_0 &= \Gamma_0 + \pi^* + \bar{\pi} + \pi_a(1-\phi_a)\theta_y, \\
\Gamma^\$ &= \Gamma + (\pi_a\phi_a, \quad \pi_u\phi_u)^\top, \\
\text{and} \quad \lambda^\$ &= \lambda + (\pi_a, \quad \pi_u)^\top.
\end{aligned}$$

The vectors λ and $\lambda^\$$ contain the real and nominal prices of risk, respectively, of productivity and policy shocks. The real price of productivity shock, $\gamma(1+x_a) - (\psi-\gamma)v_a$ is affected by the output gap, and by the continuation utility when $\psi \neq \gamma$. When prices are not flexible, monetary policy shocks have real effects and generate uncertainty in the marginal rate of substitution. As a result, there is a real compensation for policy shocks, $\gamma x_u - (\psi - \gamma)v_u$. The nominal prices of risk reflect the real prices of risk adjusted by the sensitivity of inflation to productivity and policy shocks.

Real and nominal bond yields are obtained from the pricing equation for bonds. The price of a real and nominal bonds with maturity at $t+n$ can be written as

$$\exp\left(-r_t^{(n)}\right) = \mathbb{E}_t[M_{t,t+n}], \quad \text{and} \quad \exp\left(-i_t^{(n)}\right) = \mathbb{E}_t[M_{t,t+n}^{\$}],$$

respectively, where $r_t^{(n)}$ and $i_t^{(n)}$ are the associated real and nominal bond yields, and $M_{t,t+n}$ and $M_{t,t+n}^{\$}$ are the real and nominal discount factors for payoffs at time $t+n$. It can be shown that real bond yields are

$$r_t^{(n)} = \mathcal{A}_n + \mathcal{B}_n^\top \mathbf{s}_t,$$

where the loadings satisfy the recursive equations

$$\begin{aligned} \mathcal{A}_n &= \Gamma_0 + \mathcal{A}_{n-1} + \mathcal{B}_{n-1}^\top(\mathbb{I} - \Phi)\theta_s - \frac{1}{2}(\lambda + \mathcal{B}_{n-1})^\top \Sigma(\lambda + \mathcal{B}_{n-1}), \\ \text{and} \qquad \mathcal{B}_n^\top &= \Gamma^\top + \mathcal{B}_{n-1}^\top \Phi, \end{aligned}$$

with initial conditions $\mathcal{A}_0 = 0$, and $\mathcal{B}_0 = 0$. Nominal bond yields are

$$i_t^{(n)} = \mathcal{A}_n^{\$} + \mathcal{B}_n^{\$\top} \mathbf{s}_t,$$

with coefficients $\mathcal{A}_n^{\$}$ and $\mathcal{B}_n^{\$}$ following representations similar to those of $\mathcal{A}_n$ and $\mathcal{B}_n$.

5 Analysis

This section provides an analysis of the bond pricing implications of the economic model in section 3. We analyze the short-term real rate, the term premia in real bonds and the inflation risk premia in nominal bonds. We present a baseline numerical exercise which is complemented with comparative statics for important economic parameters, and impulse responses to productivity and policy shocks. Our main interest is to understand how the real effects of monetary policy are reflected on the real term structure and the inflation risk premia. It allows us to link properties of bond yields to welfare, and the response of monetary policy to economic conditions. We focus on the volatility of real and nominal bond yields, the government borrowing costs associated to

Table 3: Baseline parameter values.

Parameter	Description	Value
β	Subjective discount factor	0.997
ψ	Inverse of EIS of consumption	0.6
γ	Coefficient of relative risk aversion	21.2
ω	Inverse of EIS of labor	0.4
α	Degree of price rigidity	0.66
θ	Elasticity of substitution of goods	5
θ_a	Average productivity growth	0.001
ϕ_a	Autocorrelation of productivity growth	0.7
σ_a	Conditional volatility of productivity growth	0.003
π^*	Inflation target	0.002
ϕ_u	Autocorrelation of policy shocks	0.9
σ_u	Conditional volatility of policy shocks	0.003
$\bar{\imath}$	Constant in the policy rule	0.0029
$\imath_\pi$	Response to inflation in the policy rule	1.1
$\imath_x$	Response to output gap in the policy rule	0.5

real bonds, and the diversification benefits of real bonds.

Table 3 presents the parameter values used in the numerical exercise. The values are standard in the macroeconomic literature, except for the coefficient of relative risk aversion γ, which under recursive preferences can be different than the inverse of the elasticity of intertemporal substitution ψ^{-1}.

Table 4 shows selected moments implied by the data and the model. The volatility of consumption and output are higher than those in the data, which implies a high volatility for a short-term nominal yield. However, it also implies a low volatility for long-term yields since the bond volatility in the model decays fast with maturity. The correlation of inflation and consumption in the data is negative, while the model implies a positive correlation.

The One-Period Real Interest Rate

The one-period real interest rate is

$$r_t = \Gamma_0 - \frac{1}{2}\lambda^\top \Sigma \lambda + \Gamma^\top \mathbf{s}_t,$$

and is affected by productivity growth and policy shocks. When prices are perfectly flexible ($\alpha = 0$), real rates are not affected by monetary policy. The response of the policy to economic

Table 4: Data and model-implied descriptive statistics 2004 − 2008.

Unconditional Moments	Monthly U.S. Data	Model
$\mathbb{E}[\Delta c]$	0.001449	0.0014
$\mathrm{sd}[\Delta c]$	0.0025	0.0085
$\mathrm{autocorr}[\Delta c]$	-0.56	0.2818
$\mathbb{E}[\pi]$	0.0029	0.0036
$\mathrm{sd}[\pi]$	0.0024	0.0059
$\mathrm{autocorr}[\pi]$	0.2411	0.8977
$\mathrm{corr}[\Delta c, \pi]$	-0.1685	0.2041
$\mathbb{E}[i^{(12)}]$	0.0031	0.0049
$\mathrm{sd}[i^{(12)}]$	0.00103	0.0029
$\mathbb{E}[r^{(24)}]$	0.0012	0.0016
$\mathrm{sd}[r^{(24)}]$	0.00084	0.000355
$\mathbb{E}[i^{(24)} - r^{(24)}]$	0.0020	0.0030

conditions and shocks in the policy rule do not have any effect on real activity. When prices are sticky, the volatility of output increases. As shown in figure 1, this increase is reflected in higher precautionary savings that lower the short-term real rate. Also, the real rate reacts not only to productivity shocks but also to policy shock. A positive policy shock reduces inflation and output and increases the real rate. A stronger response to the output gap (a higher $\imath_x$) reduces the volatility of output, reduces the precautionary savings motive and increases the real rate. Figure 2 presents additional comparative statics for the short-term rate. Lower elasticities of intertemporal substitution of consumption and labor (higher ψ and ω) increase the real rate. Risk aversion has two opposite effects on the real rate. The magnitude of the two effects depend on the coefficient of relative risk aversion γ. For low values of γ, an increase in the coefficient increases the real rate. However, as risk aversion increases, the precautionary savings effect becomes larger and reduces the real rate.

Term Premia

The average spread between long- and short-term bonds contains the average compensation for risk required by investors to hold long-term bonds over short-term bonds. It can be shown that average spreads for real bonds satisfy

$$n\mathbb{E}\left[r_t^{(n)} - r_t\right] + \frac{1}{2}\sum_{k=1}^{n-1}(n-k)^2\mathbb{E}\left[\mathrm{var}_t\left(r_{t+1}^{(n-k)}\right)\right] = \sum_{k=1}^{n-1}(n-k)\mathbb{E}\left[\mathrm{cov}_t\left(m_{t,t+1}, r_{t+1}^{(n-k)}\right)\right],$$

where

$$\sum_{k=1}^{n-1}(n-k)\mathbb{E}\left[\operatorname{cov}_t\left(m_{t,t+1}, r_{t+1}^{(n-k)}\right)\right] = -(n-1)\lambda^\top\Sigma\Gamma(\mathbb{I}-\Phi)^{-1}\left[\mathbb{I}-\Phi(\mathbb{I}-\Phi)^{-1}\left(\mathbb{I}-\frac{1}{n-1}\Phi^{(n-1)}\right)\right].$$

We can notice that

$$-\lambda^\top\Sigma\Gamma = -[\gamma(1+x_a)-(\psi-\gamma)v_a]\Gamma_a\sigma_a^2 - [\gamma x_u - (\psi-\gamma)v_u]\Gamma_u\sigma_u^2.$$

Therefore, long-term real bonds contain compensations for productivity and policy shocks. The average real term structure is downward sloping, implying negative risk premia for long-term real bonds. Negative productivity growth shocks or positive policy shocks increase the marginal rate of substitution of consumption and, simultaneously, increase the price of real bonds (lower yields). Therefore, long-term real bonds act as a hedge for consumption risk. The hedging properties of long-term bonds depend on nominal rigidities and the monetary policy rule as shown in Figure 1. Higher price rigidities increase real term premia. A stronger response to the output gap and inflation in the policy rule reduce the distortions of price rigidities in the real economy and therefore reduce term premia. Therefore, when monetary policy is more aggressive, the model implies larger hedging properties of long-term real bonds. Welfare is higher in economies with lower price rigidities or stronger policy responses to economic conditions. As a result, economies with higher welfare are characterized by long-term bonds as better vehicles to hedge consumption risk. Figure 3 shows additional comparative statics for the term premia of a two-period real bond for different model parameters. While a higher risk aversion reduces term premia, lower elasticities of intertemporal substitution of consumption and labor have the opposite effect.

Inflation Risk Premia and Government Borrowing Costs

Nominal bond yields can be decomposed into real bond yields and an inflation compensation as

$$ni_t^{(n)} = nr_t^{(n)} + \mathbb{E}[\pi_{t,t+n}] - \frac{1}{2}\operatorname{var}_t(\pi_{t,t+n}) + \operatorname{cov}_t(m_{t,t+n}, \pi_{t,t+n}),$$

where

$$\pi_{t,t+n} = \sum_{s=1}^{n}\pi_{t+s}$$

is the inflation observed between t and $t+n$ and

$$m_{t,t+n} = \sum_{s=1}^{n-1}\log M_{t+s,t+s+1}$$

is the marginal rate of substitution of consumption between t and $t+n$. The difference between nominal and real yields contains the expected inflation during the life of the bond and compensations for inflation risk or inflation risk premia. These premia capture the expected excess real return for investing in nominal n-period bonds over real n-period bonds for n periods. Investors require a compensation for holding nominal bonds over real bonds because the marginal utility of consumption is correlated with inflation and the return of nominal bonds is affected by inflation. When consumption and inflation are uncorrelated, the expected real returns on real and nominal bonds with the same maturity are the same. Then, understanding the sources of covariance between consumption and inflation helps us to understand the determinants of the inflation risk premia. It can be shown that these premia are

$$\begin{aligned}\mathrm{cov}_t(m_{t,t+n}, \pi_{t,t+n}) &= (n-1)\Gamma^\top \left[\mathbb{I} + \Phi - \mathbb{K}_n\right](\mathbb{I}-\Phi)^{-1}(\mathbb{I}-\Phi^2)^{-1}\Sigma\pi \\ &+ n\lambda^\top(\mathbb{I}-\Phi)^{-1}\left[\mathbb{I} - \frac{\Phi}{n}(\mathbb{I}-\Phi)^{-1}(\mathbb{I}-\Phi^n)\right]\Sigma\pi,\end{aligned}$$

where

$$\mathbb{K}_n = \frac{\Phi}{n-1}\left[\mathbb{I} + \Phi + \Phi(\mathbb{I}-\Phi^n)\right](\mathbb{I}-\Phi)^{-1}(\mathbb{I}-\Phi^{n-1}),$$

and $\pi = (\pi_a, \pi_u)^\top$. The sensitivities of the real discount factor and inflation to productivity growth and policy shocks determine the inflation risk premia. Since these sensitivities depend on monetary policy, it follows that the properties of risk premia are determined by the policy. Figure 1 shows the inflation risk premium of a two-period bond as a function of the degree of price rigidity and the response to economic conditions in the policy rule. Productivity growth shocks and policy shocks generate a positive covariance between inflation and consumption. A negative productivity growth shock or a positive policy shock increase the marginal utility of consumption, reduce inflation in the economy and thus increase the price of nominal bonds. Therefore, nominal bonds in this economy involve hedging properties for consumption with respect to comparable real bonds. The inflation risk premium increases as the degree of price rigidity increases, since the covariance between consumption and inflation is lower. A similar effect is observed when the policy responses to the output gap and inflation in the policy rule increases, since they involve lower distortions of shocks on inflation on output, that are reflected in a lower covariance. Since the social welfare associated to more responsive policy rules is higher, the inflation risk premium is less negative in economies with a higher social welfare.

We can link the inflation risk premium to government borrowing costs. When inflation and output are correlated the return required by investors to hold comparable real and nominal bonds differ. The government then can reduce the cost of issuing debt by choosing between real or nominal debt. Since the inflation risk premium is negative in this economy, issuing nominal bonds involves a lower financing cost for the government than issuing comparable real bonds. The savings become larger for weaker responses to economic conditions in the policy rule. When the economy is affected by policy shocks and productivity shocks, and monetary policy is conducted in such a way that welfare is low, nominal bonds involve significant savings over real bonds.

Volatility of Real and Nominal Yields

Table 1 shows that the volatility of TIPS has been similar or even higher than the volatility of comparable nominal government bonds. It is reasonable then to ask whether different monetary policies can have different implications on the volatility of real and nominal bonds. We compute the ratio of the unconditional volatilities of nominal and real bonds implied by the model. In general, this ratio is

$$\frac{\sigma(i_t^{(n)})}{\sigma(r_t^{(n)})} = \frac{\mathcal{B}_n^{\$\top}\mathrm{var}(\mathbf{s}_t)\mathcal{B}_n^{\$}}{\mathcal{B}_n^{\top}\mathrm{var}(\mathbf{s}_t)\mathcal{B}_n},$$

where $\mathrm{var}(\mathbf{s}_t) = \mathrm{diag}\left\{\frac{\sigma_a^2}{1-\phi_a^2}, \frac{\sigma_u^2}{1-\phi_u^2}\right\}$. Figure 1 shows the ratio for volatilities of one-period bonds. For low levels of price rigidities, the volatility of the short-term nominal rate is significantly higher than its real counterpart. When prices are flexible, price changes (inflation) do not generate distortions in output and do not affect real rates. Therefore volatility in inflation affects only nominal rates. As the price rigidity increases, the volatility of inflation is also reflected in volatility of real rates. Therefore, high price rigidities can explain similar volatilities of real and nominal yields. The figure also shows that more aggressive responses to economic conditions in the policy rule reduces the volatility of nominal yields with respect to real yields. The reason is that more aggressive policies reduce the volatility of output and inflation. As an extreme case, when inflation is constant, real and nominal rates move one to one. Monetary policy also affects the volatility of long-rates with respect to short-term rates. For instance, using the baseline parameter values, the volatility of a 2-year real bond with respect to the volatility of the short rate is 17%, while it is 34% for their nominal counterpart. As the response of monetary policy to economic conditions increases the rate of decay in volatility across maturities increases for both nominal and real curves.

The Correlation Between Nominal and Real Bonds and the Diversification Benefits of Real Bonds

An interesting question to ask is whether real bonds provide investors with additional diversification benefits. The complete-market environment that characterizes the economic model in this paper does not allow us to obtain a satisfactory answer to this question. However, we can provide some insights into the benefits of diversification of real bonds and how they depend on monetary policy, considering the case of a constrained investor who only has access to a nominal bond with a particular maturity. Given that there are two sources of risk affecting the marginal utility of consumption, this investor faces an incomplete market and could be benefited by the existence of a real bond. We try to capture the risk sharing benefits of a real bond for this investor computing the conditional correlation between the real return of the nominal bond and the real return of a real bond with the same maturity. Since the inflation risk premium is negative in this economy, the expected return of the real bond is lower than the the expected real return of the nominal bond. The conditional correlation at time t between real one-period returns on nominal and real bonds with maturity at $t+n$ is

$$\rho_t^{(n)} = \frac{\mathcal{B}_{n-1}^{\top}\Sigma(\mathcal{B}_{n-1}^{\$}+\pi)}{[\mathcal{B}_{n-1}^{\top}\Sigma\mathcal{B}_{n-1}]^{1/2}[(\mathcal{B}_{n}^{\$}+\pi)^{\top}\Sigma(\mathcal{B}_{n-1}^{\$}+\pi)]^{1/2}}.$$

Figure 1 shows that for low low degrees of price rigidity, this correlation decreases, implying greater benefits of diversification for the constrained investor. However, as the nominal rigidity becomes more severe, real returns on real bonds are more correlated to real returns on nominal bonds. To see this, consider the extreme case of fix prices. In this case, there is no inflation and real and nominal bonds are perfectly correlated. As the policy response to output and inflation increases, the diversification benefits of real bonds decline. Therefore, the constrained investor will be benefited by the issuance of real bonds in an economy with high volatility of inflation and distortions in output. That is, an economy with low social welfare.

6 Final Comments and Future Work

This paper explores the implications of a monetary policy with real economic effects on the compensations for risk in long-term real bonds and the inflation risk premium in nominal bonds. The analysis shows that productivity growth shocks and policy shocks imply negative compensations for risk in real bonds and negative inflation risk premia. Welfare-improving policies increase the consumption hedging properties of real bonds and imply less negative inflation risk premia, less volatile real yields and higher correlations between real returns on nominal and real bonds. Therefore, the diversification benefits of real bonds and the cost savings for issuing nominal bonds over real bonds increase when monetary policy is less responsive to the state of the economy.

We plan to incorporate to the analysis two important elements. First, the two shocks in the model economy generate a positive covariance between consumption growth and inflation, which imply a counterfactual average downward sloping nominal curve. Piazzesi and Schneider

(2007) show that, under recursive preferences, a negative covariance between consumption growth and inflation generates positive risk premia in nominal bonds. This positive covariance can be obtained by introducing markup shocks to the model. Second, compensations for risk do not vary over time in the model economy, implying constant expected returns on real and nominal bonds (the expectations hypothesis holds). We plan to introduce heteroskedasticity in policy shocks as a source of time variation in bond expected returns.

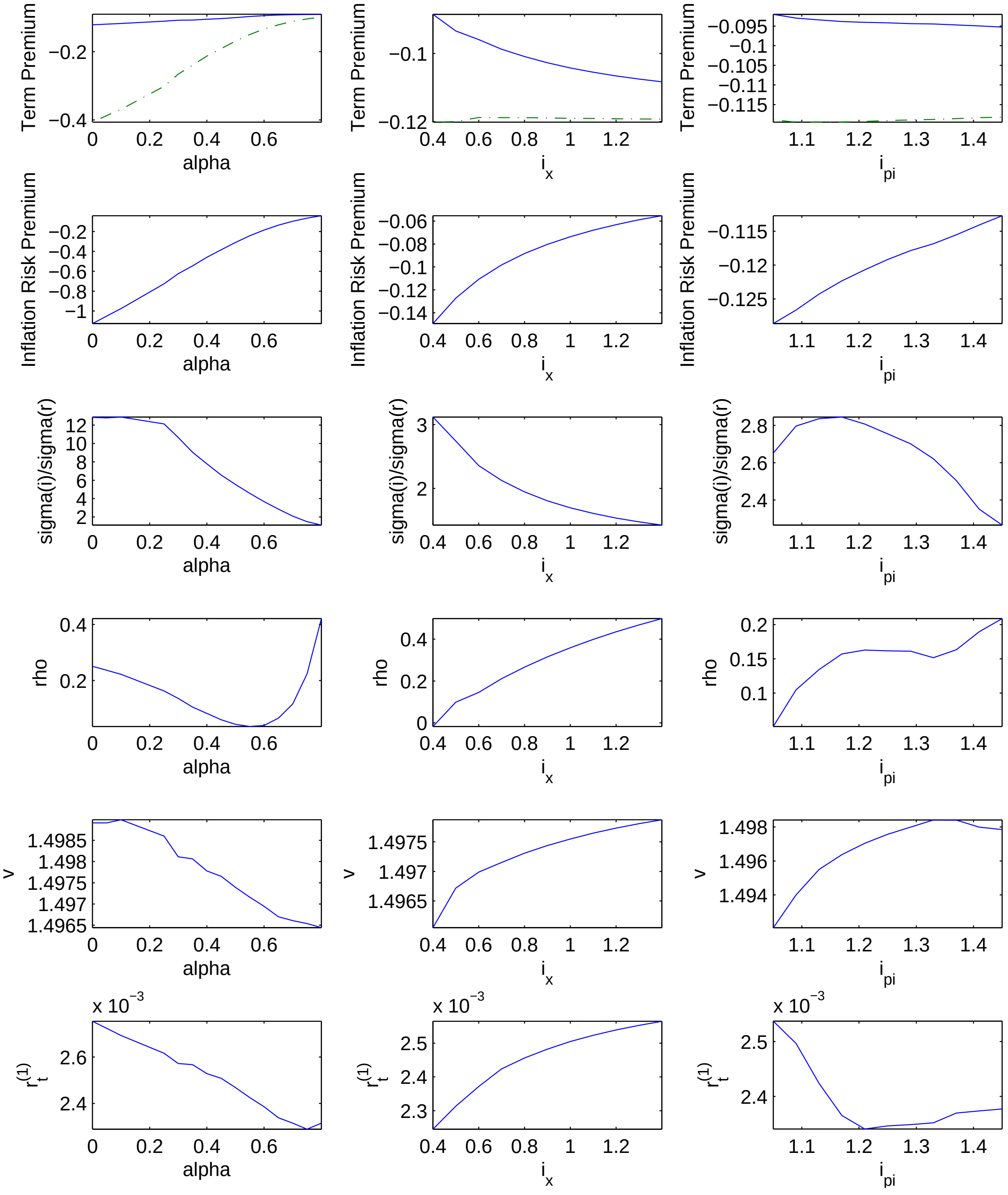

Figure 1: Comparative statics for the two-period bond term premium, the two-period bond inflation risk premium, the ratio of volatilities of one-period nominal and real bonds, the correlation between two-period real and nominal bonds (rho), the scaled continuation utility v and the one-period real rate. The parameters are the degree of nominal rigidity α, the policy response to the output gap $\imath_x$, and the policy response to inflation $\imath_\pi$.

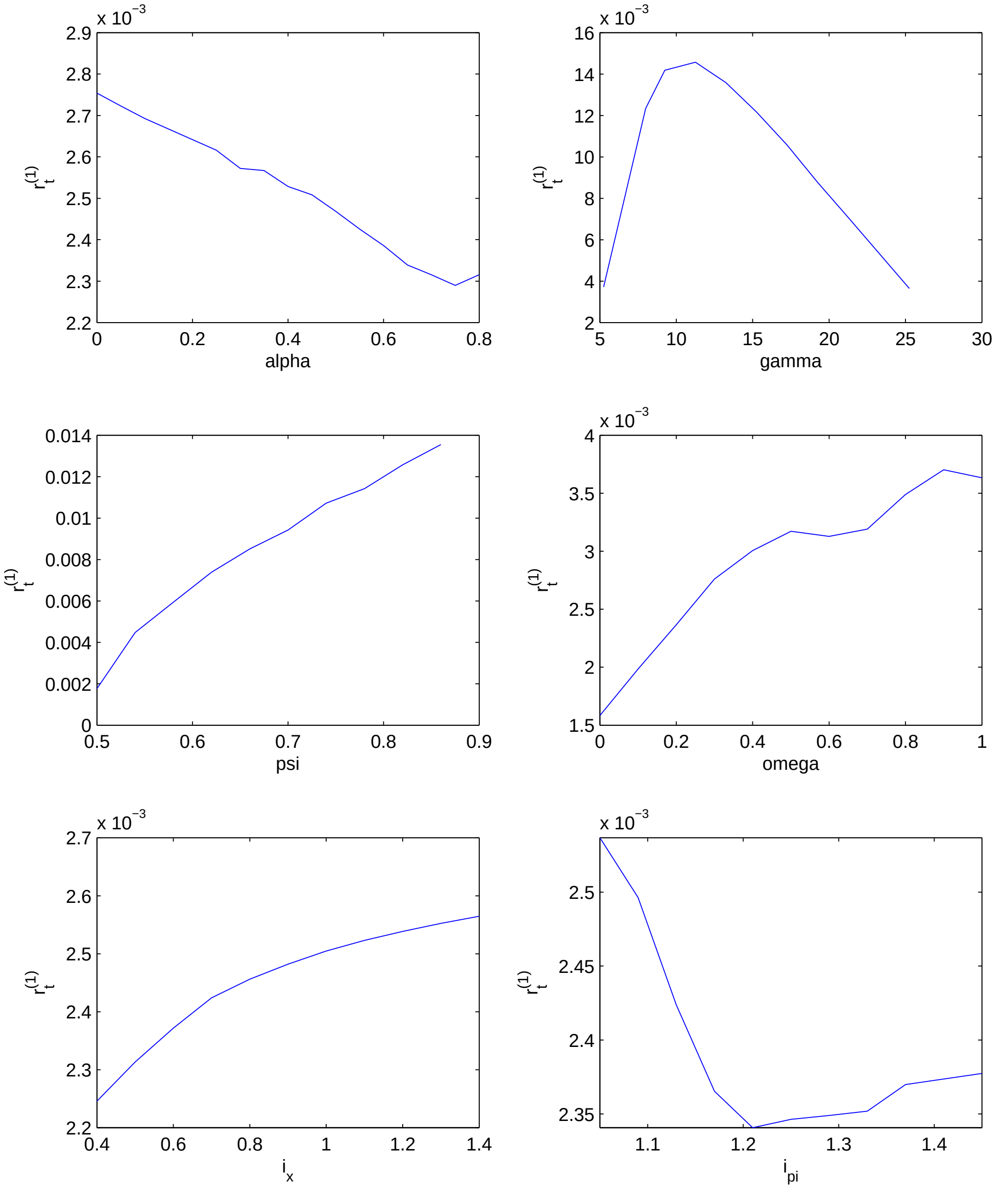

Figure 2: Comparative statics for the one-period real rate r_t and the one-period nominal rate i_t for different model parameters.

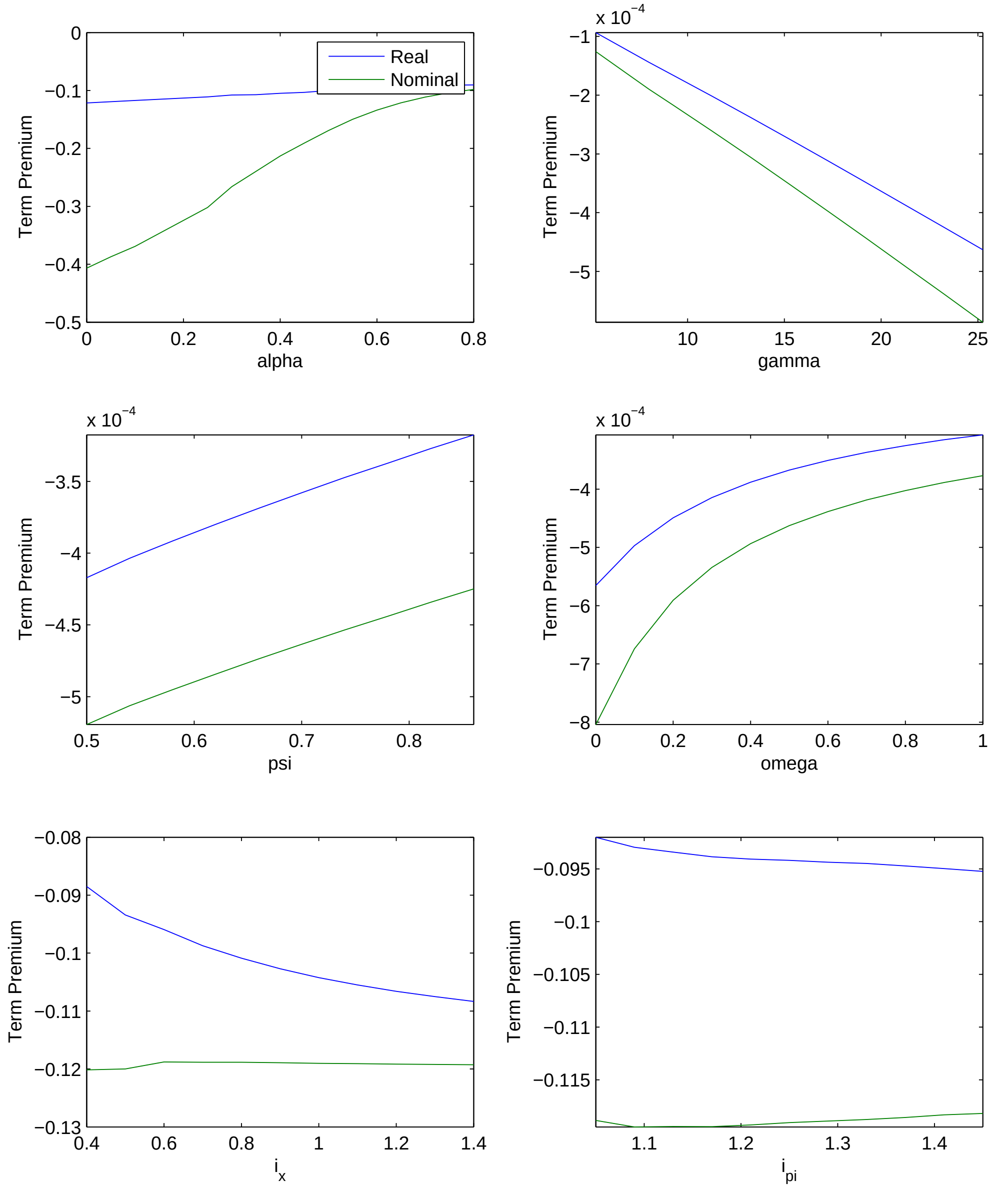

Figure 3: Comparative statics for real and nominal term premia for different model parameters. Term premia are computed as $-\Gamma^\top\Sigma\Gamma$ and $-\Gamma^{\$\top}\Sigma\Gamma^{\$}$, respectively.

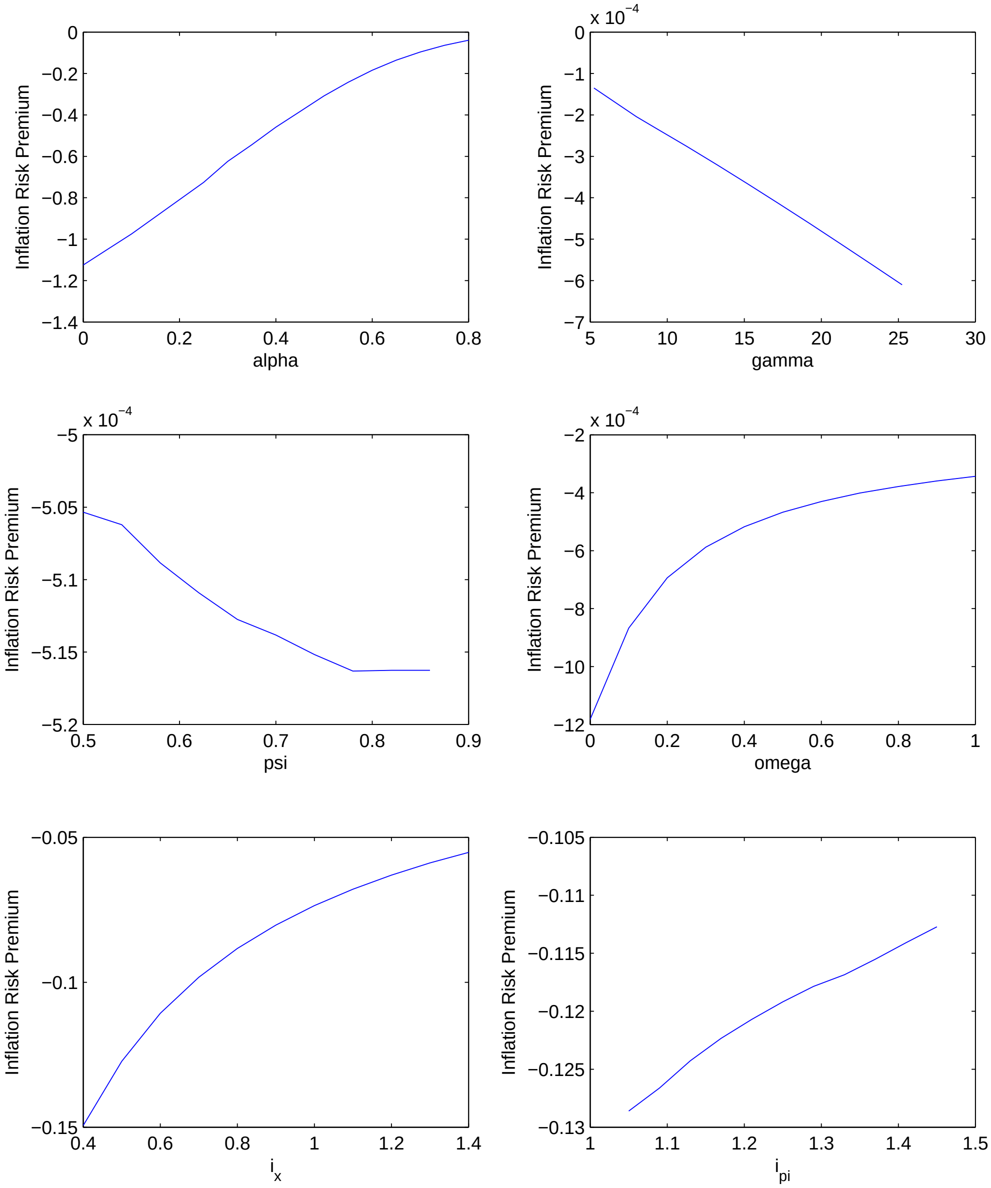

Figure 4: Comparative statics for inflation risk premia for different model parameters. Inflation risk premia are computed as $2\lambda^{\top}\Sigma\pi + \lambda^{\top}\Phi\Sigma\pi$.

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Appendix

A ECONOMIC MODEL

A.1 Households

The household's optimization problem is:

$$\begin{aligned}
\max \quad & V(C_t, N_t) = \left\{(1-\beta)U(C_t, N_t)^{1-\psi} + \beta E_t \left[V_{t+1}^{1-\gamma}\right]^{\frac{1-\psi}{1-\gamma}}\right\}^{\frac{1}{1-\psi}} \\
\text{s.t.} \quad & E_t\left[\sum_{s=0}^{\infty} M^{\$}_{t,t+s} P_{t+s} C_{t+s}\right] \leq E_t\left[\sum_{s=0}^{\infty} M^{\$}_{t,t+s}(W_{t+s}N_{t+s} + P_{t+s}\Psi_{t+s})\right] \\
\text{where} \quad & C_t = \left[\int_0^1 C_t(j)^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} \\
\text{and} \quad & U(C_t, N_t) = \left[\frac{C_t^{1-\psi}}{1-\psi} - \frac{N_t^{1+\omega}}{1+\omega}\right]^{\frac{1}{1-\psi}}.
\end{aligned}$$

The first order conditions are:

$$\frac{\partial V_t}{\partial C_t} \ : \ \frac{1}{1-\psi}\left[V_t^{1-\psi}\right]^{\frac{1}{1-\psi}-1}(1-\beta)C_t^{-\psi} - \lambda M^{\$}_{t,t} P_t = 0 \tag{8}$$

$$\frac{\partial V_t}{\partial N_t} \ : \ \frac{1}{1-\psi}\left[V_t^{1-\psi}\right]^{\frac{1}{1-\psi}-1}(1-\beta)(-N_t^{\omega}) + \lambda M^{\$}_{t,t} W_t = 0 \tag{9}$$

$$\frac{\partial V_t}{\partial C_{t+1}} \ : \ \frac{1}{1-\psi}\left[V_t^{1-\psi}\right]^{\frac{1}{1-\psi}-1}\beta\left(\frac{1-\psi}{1-\gamma}\right)E_t\left[V_{t+1}^{1-\gamma}\right]^{\frac{1-\psi}{1-\gamma}-1}(1-\gamma)V_{t+1}^{-\gamma}\frac{\partial V_{t+1}}{\partial C_{t+1}} - \lambda M^{\$}_{t,t+1}P_{t+1} = 0. \tag{10}$$

Furthermore,

$$\frac{\partial V_{t+1}}{\partial C_{t+1}} = \frac{1}{1-\psi}\left[V_{t+1}^{1-\psi}\right]^{\frac{1}{1-\psi}-1}(1-\beta)C_{t+1}^{-\psi}. \tag{11}$$

Combining (8) and (9), we have the household's intratemporal consumption and labor supply optimality condition:

$$\frac{\lambda(1-\psi)}{V_t^{\psi}(1-\beta)} = \frac{C_t^{-\psi}}{P_t} = \frac{N_t^{\omega}}{W_t} \Rightarrow \frac{W_t}{P_t} = C_t^{\psi}N_t^{\omega}.$$

Finally, combining (8) ,(10) and (11), we obtain the intertemporal consumption optimality condition:

$$\frac{\lambda(1-\psi)}{V_t^{\psi}(1-\beta)} = \frac{C_t^{-\psi}}{P_t} = \beta\left(\frac{C_{t+1}^{-\psi}}{P_{t+1}}\right)\left(\frac{V_{t+1}^{\psi-\gamma}}{M^{\$}_{t,t+1}}\right)E_t\left[V_{t+1}^{\frac{1}{1-\gamma}}\right]^{\frac{\gamma-\psi}{1-\gamma}}.$$

To get the pricing kernel, we solve for $M^{\$}_{t,t+1}$,

$$M^{\$}_{t,t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\psi}\left(\frac{P_{t+1}}{P_t}\right)^{-1}\left[\frac{V_{t+1}}{E_t[V_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}}\right]^{\psi-\gamma}. \tag{12}$$

The IS curve in this economy is

$$e^{-i_t} = E_t\left[M_{t,t+1}^{\$}\right]. \tag{13}$$

A.2 Firms

There is a dispersion of firms, denoted by j, with identical production technology in the economy. With nominal price stickiness and monopolistic competition, each firm is faced with the following optimization problem:

$$\begin{aligned}
\underset{P_t^*}{max} \quad & E_t\left[\sum_{s=0}^{\infty}\alpha^s M_{t,t+s}^{\$}\left(P_t^* Y_{t+s|t}(j) - W_{t+s|t}(j)N_{t+s|t}(j)\right)\right] && (14)\\
\text{s.t.} \quad & Y_{t+s|t}(j) = A_{t+s}N_{t+s|t}(j) && (15)\\
& Y_{t+s|t}(j) = \left(\frac{P_t^*}{P_{t+s}}\right)^{-\theta} Y_{t+s} && (16)\\
& P_t = \left[\int_0^1 P_t(j)^{1-\theta}dj\right]^{\frac{1}{1-\theta}} = \left[(1-\alpha)P_t^* + \alpha P_{t-1}\right]^{\frac{1}{1-\theta}}. && (17)
\end{aligned}$$

Using Calvo (1983) pricing, a firm can choose to optimally adjust price to P_t^* with probability α each period. Furthermore, $t+s|t$ denotes the value in period $t+s$ given that the firm last adjusted price in period t.
The first order condition for firm j is:

$$E_t\left[\sum_{s=0}^{\infty}\alpha^s M_{t,t+s}^{\$}(P_t^* Y_{t+s|t} - \mu S_{t+s|t})\right] = 0, \tag{18}$$

where $\mu = \frac{\theta}{\theta-1}$ is the frictionless markup in the absence of price adjustment constraint, and $S_{t+s|t}$ is the marginal (nominal) cost defined as:

$$S_{t+s|t} = \frac{Y_{t+s|t}^{1+\omega}}{A_{t+s}^{1+\omega}} Y_{t+s}^{\psi} P_{t+s}. \tag{19}$$

A.3 Monetary Policy

The monetary authority follows the simple Taylor rule below:

$$i_t = i + \phi_\pi \pi_t + \phi_x x_t + u_t,$$

where $u_t = \rho u_{t-1} + \sigma_u \epsilon_{u,t}$ and x_t is the output gap.

A.4 Market Clearing

$$Y_t = C_t = Y_t^F X_t.$$

B BENCHMARK: FULLY FLEXIBLE ECONOMY

B.1 Firms

Under fully flexible prices, firms in the economy maximize profit by solving the following problem:

$$\begin{aligned} &max && P_t(j)Y_t(j) - W_t(j)N_t(j) \\ &\text{s.t.} && Y_t(j) = A_t N_t(j) \\ & && Y_t(j) = \left[\frac{P_t(j)}{P_t}\right]^{-\theta} Y_t \Rightarrow P_t(j) = P_t \left[\frac{Y_t(j)}{Yt}\right]^{-\frac{1}{\theta}}. \end{aligned}$$

After substituting the constraints, the problem can be rewritten as

$$\underset{Y_t(j)}{max} \quad P_t \frac{Y_t(j)^{1-\frac{1}{\theta}}}{Y_t^{-\frac{1}{\theta}}} - W_t(j)\frac{Y_t(j)}{A_t},$$

and the accompanying first order condition is

$$P_t \left(\frac{\theta - 1}{\theta}\right) \left[\frac{Y_t(j)}{Y_t}\right]^{-\frac{1}{\theta}} - \frac{W_t(j)}{A_t} = 0.$$

Applying the definition of $P_t(j)$, we have

$$P_t(j) = \frac{W_t(j)}{A_t}\frac{\theta}{\theta - 1},$$

which can be expressed as

$$\frac{W_t(j)}{P_t} = \frac{1}{\mu}A_t = C_t^{\psi} N_t^{\omega} = Y_t^{\psi}\left(\frac{Y_t}{A_t}\right)^{\omega}, \tag{20}$$

where $\mu = \frac{\theta}{1-\theta}$ denotes the fully flexible markup by the firm. The second equality comes from the household's optimality condition, and the third equality comes from the firm's production function. Furthermore, since the firms are homogeneous, $P_t = P_t(j)$, and $Y_t = C_t$ since there is no investment and government spending. Finally, the fully flexible output in the economy is

$$Y_t^{F^{\psi+\omega}} = \frac{1}{\mu}A_t^{1+\omega},$$

or in logs:

$$\Delta y_{t+1}^F = \frac{1+\omega}{\psi+\omega}\Delta a_{t+1}. \tag{21}$$

C Sticky Price Economy

C.1 Households

The household's utility is:

$$U(C_t, N_t) = \frac{C_t^{1-\psi}}{1-\psi} - e^{\chi_t}\frac{N_t^{1+\omega}}{1+\omega},$$

where $\chi_{t+1} = (1-\phi_\chi)\theta_\chi + \phi_\chi \chi_t + \sigma_\chi \sqrt{1 + Z_{\chi,t} I_\chi} \epsilon_{\chi,t+1}$. The resulting marginal rate of substitution between labor and consumption is:

$$\frac{W_t(j)}{P_t} = e^{\chi_t} C_t^{\psi} N_t^{\omega}. \tag{22}$$

C.2 Fully Flexible Output

From (20),

$$\frac{W_t(j)}{P_t} = \frac{W_t}{P_t} = \frac{1}{\mu} A_t,$$

and substitute in (22),

$$\begin{aligned}
& \frac{1}{\mu} A_t = e^{\chi_t} Y_t^{\psi} N_t^{\omega} \\
\Rightarrow \quad & Y_t^{\psi+\omega} = \frac{e^{-\chi_t}}{\mu} A_t^{1+\omega} \\
\Rightarrow \quad & \Delta y_t = \frac{1}{\psi+\omega} [(1+\omega)\Delta a_t - \Delta \chi_t].
\end{aligned}$$

C.3 Firms

There is a dispersion of firms, denoted by j, with identical production technology in the economy. With nominal price stickiness and monopolistic competition, each firm's price setting behavior maximizes the following profit function:

$$\begin{aligned}
\underset{P_t^*}{max} \quad & E_t \left[\sum_{s=0}^{\infty} \alpha^s M^{\$}_{t,t+s} \left(P_t^* (\pi^*)^s Y_{t+s|t}(j) - W_{t+s|t}(j) N_{t+s|t}(j) \right) \right] \\
\text{s.t.} \quad & Y_{t+s|t}(j) = A_{t+s} N_{t+s|t}(j) \\
& Y_{t+s|t}(j) = \left(\frac{P_t^* (\pi^*)^s}{P_{t+s}} \right)^{-\theta} Y_{t+s}.
\end{aligned}$$

Using Calvo (1983) pricing, a firm can choose to optimally adjust price to P_t^* with probability α each period. If a firm does not adjust its price in a certain period, the price is indexed to inflation, π^*. As before, $t+s|t$ denotes the value in period $t+s$ given that the firm last adjusted price in period t.

The first order condition for firm j is:

$$\begin{aligned}
& E_t \left[\sum_{s=0}^{\infty} \alpha^s M^{\$}_{t,t+s} \left((1-\theta)(\pi^*) Y_{t+s|t} - \frac{W_{t+s|t}}{A_{t+s}} (-\theta) \frac{Y_{t+s|t}}{P_t^*} \right) \right] \\
\Rightarrow \quad & E_t \left[\sum_{s=0}^{\infty} \alpha^s M^{\$}_{t,t+s} Y_{t+s|t} \left(P_t^* (\pi^*)^s - \mu \frac{W_{t+s|t}}{A_{t+s}} \right) \right] = 0 \\
\Rightarrow \quad & E_t \left[\sum_{s=0}^{\infty} \alpha^s M^{\$}_{t,t+s} Y_{t+s|t} \left(P_t^* (\pi^*)^s - \mu e^{\chi_{t+s}} P_{t+s} Y^{\psi}_{t+s} \frac{Y^{\omega}_{t+s|t}}{A^{1+\omega}_{t+s}} \right) \right] = 0
\end{aligned}$$

where is the frictionless markup as defined before, and the last equality comes uses the definition, $\frac{W_{t+s|t}}{P_{t+s}} = e^{\chi_{t+s}} Y_{t+s}^{\psi} N_{t+s|t}^{\omega} = e^{\chi_{t+s}} Y_{t+s}^{\psi} \frac{Y_{t+s|t}^{\omega}}{A_{t+s}^{\omega}}$. Replace $Y_{t+s|t}$ by $\left(\frac{P_t^*(\pi^*)^s}{P_{t+s}}\right)^{-\theta} Y_{t+s}$, the first order condition can be rewritten as the following,

$$\underbrace{E_t\left[\sum_{s=0}^{\infty}(\alpha\pi^*)^s M_{t,t+s}^{\$} Y_{t+s}\left(\frac{P_t^*(\pi^*)^s}{P_{t+s}}\right)^{-\theta} P_t^*\right]}_{L.H.S.} = \underbrace{E_t\left[\sum_{s=0}^{\infty}\alpha^s M_{t,t+s}^{\$}\mu e^{\chi_{t+s}}\left(\frac{1}{A_{t+s}}\right)^{1+\omega} Y_{t+s}^{1+\omega+\psi}\left(\frac{P_t^*(\pi^*)^s}{P_{t+s}}\right)^{-\theta(1+\omega)} P_{t+s}\right]}_{R.H.S},$$

where

$$L.H.S. = P_t^*\left(\frac{P_t^*}{P_t}\right)^{-\theta} Y_t\, \underbrace{E_t\left[\sum_{s=0}^{\infty}(\alpha\pi^*)^s M_{t,t+s}^{\$}\frac{Y_{t+s}}{Y_t}\left(\frac{P_t(\pi^*)^s}{P_{t+s}}\right)^{-\theta}\right]}_{H_t}$$

and

$$R.H.S. = \mu Y_t^{1+\omega+\psi}\left(\frac{P_t^*}{P_t}\right)^{-\theta(1+\omega)}\left(\frac{P_t}{A_t^{1+\omega}}\right) e^{\chi_t}\, \underbrace{E_t\left[\sum_{s=0}^{\infty}\alpha^s M_{t,t+s}^{\$}\frac{e^{\chi_{t+s}}}{e^{\chi_t}}\left(\frac{A_t}{A_{t+s}}\right)^{1+\omega}\left(\frac{Y_{t+s}}{Y_t}\right)^{1+\omega+\psi}\left(\frac{P_t(\pi^*)^s}{P_{t+s}}\right)^{-\theta(1+\omega)}\frac{P_{t+s}}{P_t}\right]}_{G_t}.$$

Taking H_t, it can be recursively written such that,

$$\begin{aligned}
H_t &= 1 + E_t\left[\sum_{s=1}^{\infty}(\alpha\pi^*)^s M_{t,t+1}^{\$} M_{t+1,t+s}^{\$}\left(\frac{Y_{t+s}}{Y_{t+1}}\right)\left(\frac{Y_{t+1}}{Y_t}\right)\left(\frac{P_t\pi^*}{P_{t+1}}\right)^{-\theta}\left(\frac{P_{t+1}(\pi^*)^{s-1}}{P_{t+s}}\right)^{-\theta}\right] \\
&= 1 + \alpha\pi^* E_t\left[\underbrace{M_{t,t+1}^{\$}\left(\frac{P_t\pi^*}{P_{t+1}}\right)^{-\theta}\left(\frac{Y_{t+1}}{Y_t}\right)}_{\Lambda_{t,t+1}^P} E_{t+1}\left[\sum_{s=1}^{\infty}(\alpha\pi^*)^{s-1} M_{t+1,t+s}^{\$}\left(\frac{Y_{t+s}}{Y_{t+1}}\right)\left(\frac{P_{t+1}(\pi^*)^{s-1}}{P_{t+s}}\right)^{-\theta}\right]\right] \\
&= 1 + \alpha\pi^* E_t\left[\Lambda_{t,t+1}^P H_{t+1}\right].
\end{aligned}$$

Similarly, G_t has the following recursive formulation:

$$\begin{aligned}
G_t &= 1 + E_t\left[\sum_{s=1}^{\infty}\alpha^s M_{t,t+1}^{\$} M_{t+1,t+s}^{\$}\frac{e^{\chi_{t+s}}}{e^{\chi_{t+1}}}\frac{e^{\chi_{t+1}}}{e^{\chi_t}}\left(\frac{A_t}{A_{t+1}}\right)^{1+\omega}\left(\frac{A_{t+1}}{A_{t+s}}\right)^{1+\omega}\left(\frac{Y_{t+s}}{Y_{t+1}}\right)^{1+\omega+\psi}\left(\frac{Y_{t+1}}{Y_t}\right)^{1+\omega+\psi}\right. \\
&\qquad\left.\left(\frac{P_t\pi^*}{P_{t+1}}\right)^{-\theta(1+\omega)}\left(\frac{P_{t+1}(\pi^*)^{s-1}}{P_{t+s}}\right)^{-\theta(1+\omega)}\frac{P_{t+s}}{P_{t+1}}\frac{P_{t+1}}{P_t}\right] \\
&= 1 + \alpha E_t\left[\underbrace{M_{t,t+1}^{\$}\frac{e^{\chi_{t+1}}}{e^{\chi_t}}\left(\frac{A_t}{A_{t+1}}\right)^{1+\omega}\left(\frac{P_t\pi^*}{P_{t+1}}\right)^{-\theta(1+\omega)}\left(\frac{Y_{t+1}}{Y_t}\right)^{1+\omega+\psi}\frac{P_{t+1}}{P_t}}_{\Lambda_{t,t+1}^S}\right. \\
&\qquad\left. E_{t+1}\left[\sum_{s=1}^{\infty}\alpha^{s-1} M_{t+1,t+s}^{\$}\frac{e^{\chi_{t+s}}}{e^{\chi_{t+1}}}\left(\frac{A_{t+1}}{A_{t+s}}\right)^{1+\omega}\left(\frac{Y_{t+s}}{Y_{t+1}}\right)^{1+\omega+\psi}\left(\frac{P_{t+1}(\pi^*)^{s-1}}{P_{t+s}}\right)^{-\theta(1+\omega)}\frac{P_{t+s}}{P_{t+1}}\right]\right] \\
&= 1 + \alpha E_t\left[\Lambda_{t,t+1}^S G_{t+1}\right].
\end{aligned}$$

D Taylor Rule Approach

For this section, assume $\chi_t = 0$ for $\forall\ t$.

D.1 System

$$e^{-i_t} = E_t[M^{\$}_{t,t+1}]. \tag{23}$$

$$\left(\frac{P_t^*}{P_t}\right)^{1+\theta\omega} H_t = X_t^{\omega+\psi} G_t \tag{24}$$

$$\frac{P_t^*}{P_t} = \left[\frac{1}{1-\alpha}\left(1-\alpha\left(\frac{\Pi^*}{\Pi_t}\right)^{1-\theta}\right)\right]^{\frac{1}{1-\theta}} \tag{25}$$

$$H_t = 1 + \alpha\Pi^* E_t\left[M^{\$}_{t,t+1}\frac{Y^F_{t+1}}{Y^F_t}\frac{X_{t+1}}{X_t}\left(\frac{\Pi^*}{\Pi_{t+1}}\right)^{-\theta} H_{t+1}\right], \tag{26}$$

and

$$G_t = 1 + \alpha\Pi^* E_t\left[M^{\$}_{t,t+1}\frac{Y^F_{t+1}}{Y^F_t}\left(\frac{X_{t+1}}{X_t}\right)^{1+\omega+\psi}\left(\frac{\Pi^*}{\Pi_{t+1}}\right)^{-1-\theta(1+\omega)} G_{t+1}\right], \tag{27}$$

where

$$M^{\$}_{t,t+1} = \beta\left(\frac{Y^F_{t+1}}{Y^F_t}\right)^{-\psi}\left(\frac{X_{t+1}}{X_t}\right)^{-\psi}\left(\frac{V_{t+1}}{E_t\left[V_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\psi-\gamma}\left(\Pi_{t+1}\right)^{-1}. \tag{28}$$

The simple Taylor rule is $i_t = \bar{i} + i_x X_t + i_\Pi \Pi_t + u_t$, where

$$u_{t+1} = \phi_u u_t + \sigma_u \epsilon_{u,t+1}.$$

The fully flexible output in the economy is $\Delta y^F_t = \frac{1+\omega}{\omega+\psi}\Delta a_t$, where

$$\Delta a_{t+1} = (1-\phi_a)\theta_a + \phi_a \Delta a_t + \sigma_a \epsilon_{a,t+1}.$$

The process for output growth, thus, can be rewritten as:

$$\Delta y^F_{t+1} = (1-\phi_a)\theta_y + \phi_a \Delta y^F_t + \sigma_y \epsilon_{a,t+1},$$

where $\theta_y = \frac{1+\omega}{\omega+\psi}\theta_a$, and $\sigma_y = \frac{1+\omega}{\omega+\psi}\sigma_a$.

D.2 Log Linearization

Combining (24) and (25), we have

$$\begin{aligned}
&\left[\frac{1}{1-\alpha}\left(1-\alpha\left(\frac{\Pi^*}{\Pi_t}\right)^{1-\theta}\right)\right]^{\frac{1+\theta\omega}{1-\theta}} H_t = X_t^{\omega+\psi} G_t \\
\Rightarrow\quad & \frac{1+\theta\omega}{1-\theta} log\left[\frac{1}{1-\alpha}\left(1-\alpha e^{(1-\theta)(\pi^*-\pi_t)}\right)\right] + h_t = (\omega+\psi)x_t + g_t \\
\Rightarrow\quad & \left(\frac{1+\theta\omega}{1-\theta}\right)\left[D_\pi + F_\pi(\pi_t - \overline{m_\pi})\right] + h_t = (\omega+\psi)x_t + g_t,
\end{aligned}$$

incorporating the log linear approximation of π_t around $\overline{m_\pi}$ in the following:

$$\begin{aligned}
& log\left[\frac{1}{1-\alpha}\left(1-\alpha e^{(1-\theta)(\pi^*-\pi_t)}\right)\right] \\
\approx\quad & \underbrace{log\left[\frac{1-\alpha e^{-(1-\theta)(\overline{m_\pi}-\pi^*)}}{1-\alpha}\right]}_{D_\pi} + \underbrace{\frac{\alpha(1-\theta)e^{-(1-\theta)(\overline{m_\pi}-\pi^*)}}{1-\alpha e^{-(1-\theta)(\overline{m_\pi}-\pi^*)}}}_{F_\pi}(\pi_t - \overline{m_\pi}) \\
=\quad & D_\pi + F_\pi(\pi_t - \overline{m_\pi}),
\end{aligned}$$

where $\overline{m_\pi} = E[\pi_t] = \pi^* + \bar{\pi} + pi_a\theta_y$.

To find log linearized value function, we start with the one period utility:

$$\begin{aligned}
U(C_t, N_t)^{1-\psi} &= \frac{C_t^{1-\psi}}{1-\psi} - \int_0^1 \frac{N_t(j)^{1+\omega}}{1+\omega} dj \\
&= C_t^{1-\psi}\left[\frac{1}{1-\psi} - \frac{1}{1-\mu}\frac{1}{1+\omega} X_t^{\psi+\omega}\int_0^1\left(\frac{P_t(j)}{P_t}\right)^{-\theta(1+\omega)} dj\right]
\end{aligned}$$

Define $F_t = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\theta(1+\omega)} dj$, it can be rewritten as:

$$\begin{aligned}
F_t &= \int_{j\in[1-\alpha]}\left(\frac{P_t^*}{P_t}\right)^{-\theta(1+\omega)} dj + \int_{j\in[\alpha]}\left(\frac{P_{t-1}(j)\Pi^*}{P_t}\right)^{-\theta(1+\omega)} dj \\
&= (1-\alpha)\left(\frac{P_t^*}{P_t}\right)^{-\theta(1+\omega)} + \alpha\left(\frac{\Pi^*}{\Pi_t}\right)^{-\theta(1+\omega)}\int_0^1\left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\theta(1+\omega)} dj \\
&= (1-\alpha)\left(\frac{P_t^*}{P_t}\right)^{-\theta(1+\omega)} + \alpha\left(\frac{\Pi^*}{\Pi_t}\right)^{-\theta(1+\omega)} F_{t-1}.
\end{aligned}$$

Substitute in (25), we have

$$F_t = (1-\alpha)\left[\frac{1}{1-\alpha}\left(1-\alpha\left(\frac{\Pi^*}{\Pi_t}\right)^{1-\theta}\right)\right]^{\frac{-\theta(1+\omega)}{1-\theta}} + \alpha\left(\frac{\Pi^*}{\Pi_t}\right)^{-\theta(1+\omega)} F_{t-1},$$

which then can be linearized around $\pi_t = \pi^*$ and $f_{t-1} = m_f$, where $e^{m_f} = \frac{1-\alpha}{1-\alpha} = 1$ since in steady state,

$F_t = F_{t-1} = M_f$. Therefore, $F_t = 1$.

Again, the Epstein-Zin utility is of the following form:

$$V_t^{1-\psi} = (1-\beta)U(C_t, N_t)^{1-\psi} + \beta E_t \left[V_{t+1}^{1-\gamma}\right]^{\frac{1-\psi}{1-\gamma}}.$$

Let $v_t \equiv log\frac{V_t}{C_t}$, the recursive utility in log form is:

$$e^{(1-\psi)v_t} = (1-\beta)\left[\frac{1}{1-\psi} - \frac{1}{\mu(1+\omega)}e^{(\psi+\omega)x_t}\right] + \beta exp\left\{\left(\frac{1-\psi}{1-\gamma}\right)\underbrace{logE_t\left[e^{(1-\gamma)(v_{t+1}+\Delta y^F_{t+1}+\Delta x_{t+1})}\right]}_{\Upsilon_t}\right\}.$$

Taking logs on both sides, we then linearize around $x_t = \bar{x}$ and $\Upsilon_t = \bar{\Upsilon}$:

$$\begin{aligned}
(1-\psi)v_t &= log\left[(1-\beta)\left[\frac{1}{1-\psi} - \frac{1}{\mu(1+\omega)}e^{(\psi+\omega)x_t}\right] + \beta exp\left\{\left(\frac{1-\psi}{1-\gamma}\right)\Upsilon_t\right\}\right] && (29)\\
&\approx log\left[\underbrace{(1-\beta)\left[\frac{1}{1-\psi} - \frac{1}{\mu(1+\omega)}e^{(\psi+\omega)\bar{x}}\right] + \beta exp\left\{\left(\frac{1-\psi}{1-\gamma}\right)\bar{\Upsilon}\right\}}_{D_v}\right] - && (30)\\
&\qquad \frac{(1-\beta)}{\mu(1+\omega)}\frac{(\psi+\omega)}{D_v}e^{(\psi+\omega)\bar{x}}(x_t-\bar{x}) + \frac{\beta}{D_v}\left(\frac{1-\psi}{1-\gamma}\right)e^{\left(\frac{1-\psi}{1-\gamma}\right)\bar{\Upsilon}}(\Upsilon - \bar{\Upsilon}) && (31)\\
&= \underbrace{logD_v + \frac{(1-\beta)}{\mu(1+\omega)}\frac{(\psi+\omega)}{D_v}e^{(\psi+\omega)\bar{x}}\bar{x} - \frac{\beta}{D_v}\left(\frac{1-\psi}{1-\gamma}\right)e^{\left(\frac{1-\psi}{1-\gamma}\right)\bar{\Upsilon}}\bar{\Upsilon}}_{\overline{\eta_v}} && (32)\\
&\qquad \underbrace{-\frac{(1-\beta)}{\mu(1+\omega)}\frac{(\psi+\omega)}{D_v}e^{(\psi+\omega)\bar{x}}}_{\eta_{vx}}x_t + \underbrace{\frac{\beta}{D_v}\left(\frac{1-\psi}{1-\gamma}\right)e^{\left(\frac{1-\psi}{1-\gamma}\right)\bar{\Upsilon}}}_{\eta_{vv}}\Upsilon_t && (33)\\
&= \overline{\eta_v} + \eta_{vx}x_t + \eta_{vv}logE_t\left[e^{(1-\gamma)(v_{t+1}+\Delta y^F_{t+1}+\Delta x_{t+1})}\right], && (34)
\end{aligned}$$

where the second line is the result of a first order Taylor expansion around the steady states.

$$\begin{aligned}
\bar{\Upsilon} &= -(1-\gamma)x_a\theta_y + (1-\gamma)\bar{v} + (1-\gamma)(1+v_a+x_a)\theta_y + \\
&\quad \frac{1}{2}(1-\gamma)^2(1+v_a+x_a)^2\sigma_y^2 + \frac{1}{2}(1-\gamma)^2(v_u+x_u)^2\sigma_u^2
\end{aligned}$$

To linearize H_t, we combine (26) and (28):

$$e^{h_t} = 1 + \alpha\beta E_t\left[\left(\frac{Y^F_{t+1}}{Y^F_t}\right)^{1-\psi}\left(\frac{X_{t+1}}{X_t}\right)^{1-\psi}\left(\frac{\Pi^*}{\Pi_{t+1}}\right)^{1-\theta}\left(\frac{V_{t+1}}{E_t\left[V_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\psi-\gamma}H_{t+1}\right].$$

Note:

$$\frac{V_{t+1}}{E_t\left[V_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}} = \frac{e^{v_{t+1}+\Delta y^F_{t+1}+\Delta x_{t+1}}}{e^{\frac{1}{1-\gamma} logE_t\left[e^{(1-\gamma)(v_{t+1}+\Delta y^F_{t+1}+\Delta x_{t+1})}\right]}},$$

and from (34)

$$logE_t\left[e^{(1-\gamma)(v_{t+1}+\Delta y^F_{t+1}+\Delta x_{t+1})}\right] = \frac{1}{\eta_{vv}}\left[(1-\psi)v_t - \overline{\eta_v} - \eta_{vx}x_t\right],$$

therefore,

$$\begin{aligned}
& logE_t\left[\left(\frac{Y^F_{t+1}}{Y^F_t}\right)^{1-\psi}\left(\frac{X_{t+1}}{X_t}\right)^{1-\psi}\left(\frac{\Pi^*}{\Pi_{t+1}}\right)^{1-\theta}\left(\frac{V_{t+1}}{E_t\left[V_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\psi-\gamma} H_{t+1}\right] \\
= \;& logE_t\left[e^{(1-\psi)\Delta y^F_{t+1}+(1-\psi)\Delta x_{t+1}+(1-\theta)(\pi^*-\pi_{t+1})}\frac{e^{(\psi-\gamma)(v_{t+1}+\Delta y^F_{t+1}+\Delta x_{t+1})}}{e^{\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)v_t-\overline{\eta_v}-\eta_{vx}x_t\right]}}H_{t+1}\right] \\
= \;& logE_t\left[e^{(1-\gamma)\Delta y^F_{t+1}+(1-\gamma)\Delta x_{t+1}+(1-\theta)(\pi^*-\pi_{t+1})}\frac{e^{(\psi-\gamma)v_{t+1}}}{e^{\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)v_t-\overline{\eta_v}-\eta_{vx}x_t\right]}}H_{t+1}\right] \\
= \;& logE_t\left[e^{(1-\gamma)(\Delta y^F_{t+1}+\Delta x_{t+1})+(1-\theta)(\pi^*-\pi_{t+1})+(\psi-\gamma)v_{t+1}+h_{t+1}}e^{-\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)v_t-\overline{\eta_v}-\eta_{vx}x_t\right]}\right] \\
= \;& logE_t\left[e^{(1-\gamma)(\Delta y^F_{t+1}+\Delta x_{t+1})+(1-\theta)(\pi^*-\pi_{t+1})+(\psi-\gamma)v_{t+1}+h_{t+1}}\right] - \frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)v_t - \overline{\eta_v} - \eta_{vx}x_t\right] \\
= \;& \Xi_t.
\end{aligned}$$

Now, log linearized around $\Xi_t = \bar{\Xi}$:

$$\begin{aligned}
h_t \;=\;& log\left[1+\alpha\beta e^{\Xi_t}\right] \\
\approx\;& log\left[1+\alpha\beta e^{\bar{\Xi}}\right] + \frac{\alpha\beta e^{\bar{\Xi}}}{1+\alpha\beta e^{\bar{\Xi}}}(\Xi_t - \bar{\Xi}) \\
=\;& log\left[1+\alpha\beta e^{\bar{\Xi}}\right] - \frac{\alpha\beta e^{\bar{\Xi}}}{1+\alpha\beta e^{\bar{\Xi}}}\bar{\Xi} - \frac{\alpha\beta e^{\bar{\Xi}}}{1+\alpha\beta e^{\bar{\Xi}}}\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)v_t - \overline{\eta_v} - \eta_{vx}x_t\right] + \\
& \frac{\alpha\beta e^{\bar{\Xi}}}{1+\alpha\beta e^{\bar{\Xi}}} logE_t\left[e^{(1-\gamma)(\Delta y^F_{t+1}+\Delta x_{t+1})+(1-\theta)(\pi^*-\pi_{t+1})+(\psi-\gamma)v_{t+1}+h_{t+1}}\right] \\
=\;& \underbrace{log\left[1+\alpha\beta e^{\bar{\Xi}}\right] - \frac{\alpha\beta e^{\bar{\Xi}}}{1+\alpha\beta e^{\bar{\Xi}}}\bar{\Xi} + \frac{\alpha\beta e^{\bar{\Xi}}}{1+\alpha\beta e^{\bar{\Xi}}}\frac{\psi-\gamma}{1-\gamma}\frac{\overline{\eta_v}}{\eta_{vv}}}_{\overline{\eta_h}} \underbrace{-\frac{\alpha\beta e^{\bar{\Xi}}}{1+\alpha\beta e^{\bar{\Xi}}}\frac{\psi-\gamma}{1-\gamma}\frac{1-\psi}{\eta_{vv}}}_{\eta_{hv}} v_t + \\
& \underbrace{\frac{\alpha\beta e^{\bar{\Xi}}}{1+\alpha\beta e^{\bar{\Xi}}}\frac{\psi-\gamma}{1-\gamma}\frac{\eta_{vx}}{\eta_{vv}}}_{\eta_{hx}} x_t + \underbrace{\frac{\alpha\beta e^{\bar{\Xi}}}{1+\alpha\beta e^{\bar{\Xi}}}}_{\eta_{hh}} logE_t\left[e^{(1-\gamma)(\Delta y^F_{t+1}+\Delta x_{t+1})+(1-\theta)(\pi^*-\pi_{t+1})+(\psi-\gamma)v_{t+1}+h_{t+1}}\right] \\
=\;& \overline{\eta_h} + \eta_{hv}v_t + \eta_{hx}x_t + \eta_{hh} logE_t\left[e^{(1-\gamma)(\Delta y^F_{t+1}+\Delta x_{t+1})+(1-\theta)(\pi^*-\pi_{t+1})+(\psi-\gamma)v_{t+1}+h_{t+1}}\right],
\end{aligned}$$

where line two employs the first order Taylor series approximation.

Furthermore,

$$\begin{aligned}\bar{\Xi} \;=\;& -(1-\gamma)x_a\theta_y - \bar{\pi} + (\psi-\gamma)\bar{v} + \bar{h} + \left[(1-\gamma)(1+x_a) - \pi_a + (\psi-\gamma)v_a + h_a\right]\theta_y + \\ & \frac{1}{2}\left[(1-\gamma)(1+x_a) - \pi_a + (\psi-\gamma)v_a + h_a\right]^2 \sigma_y^2 + \frac{1}{2}\left[(1-\gamma)x_u - \pi_u + (\psi-\gamma)v_u + h_u\right]^2 \sigma_u^2 - \\ & \frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)\bar{v} - \overline{\eta_v} - \eta_{vx}\bar{x} + ((1-\psi)v_a - \eta_{vx}x_a)\theta_y\right]\end{aligned}$$

To linearize G_t, we combine (27) and (28):

$$e^{g_t} = 1 + \alpha\beta E_t\left[\left(\frac{Y^F_{t+1}}{Y^F_t}\right)^{1-\psi}\left(\frac{X_{t+1}}{X_t}\right)^{1+\omega}\left(\frac{\Pi^*}{\Pi_{t+1}}\right)^{-\theta(1+\omega)}\left(\frac{V_{t+1}}{E_t\left[V_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\psi-\gamma} G_{t+1}\right].$$

Note:

$$\frac{V_{t+1}}{E_t\left[V_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}} = \frac{e^{v_{t+1}+\Delta y^F_{t+1}+\Delta x_{t+1}}}{e^{\frac{1}{1-\gamma}logE_t\left[e^{(1-\gamma)(v_{t+1}+\Delta y^F_{t+1}+\Delta x_{t+1})}\right]}},$$

and from (34)

$$logE_t\left[e^{(1-\gamma)(v_{t+1}+\Delta y^F_{t+1}+\Delta x_{t+1})}\right] = \frac{1}{\eta_{vv}}\left[(1-\psi)v_t - \overline{\eta_v} - \eta_{vx}x_t\right],$$

therefore,

$$\begin{aligned} & logE_t\left[\left(\frac{Y^F_{t+1}}{Y^F_t}\right)^{1-\psi}\left(\frac{X_{t+1}}{X_t}\right)^{1+\omega}\left(\frac{\Pi^*}{\Pi_{t+1}}\right)^{-\theta(1+\omega)}\left(\frac{V_{t+1}}{E_t\left[V_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\psi-\gamma} G_{t+1}\right] \\ = \;& logE_t\left[e^{(1-\psi)\Delta y^F_{t+1}+(1+\omega)\Delta x_{t+1}-\theta(1+\omega)(\pi^*-\pi_{t+1})}\frac{e^{(\psi-\gamma)(v_{t+1}+\Delta y^F_{t+1}+\Delta x_{t+1})}}{e^{\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}[(1-\psi)v_t-\overline{\eta_v}-\eta_{vx}x_t]}}G_{t+1}\right] \\ = \;& logE_t\left[e^{(1-\gamma)\Delta y^F_{t+1}+(1+\omega+\psi-\gamma)\Delta x_{t+1}-\theta(1+\omega)(\pi^*-\pi_{t+1})}\frac{e^{(\psi-\gamma)v_{t+1}}}{e^{\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}[(1-\psi)v_t-\overline{\eta_v}-\eta_{vx}x_t]}}G_{t+1}\right] \\ = \;& logE_t\left[e^{(1-\gamma)\Delta y^F_{t+1}+(1+\omega+\psi-\gamma)\Delta x_{t+1}-\theta(1+\omega)(\pi^*-\pi_{t+1})+(\psi-\gamma)v_{t+1}+g_{t+1}}e^{-\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}[(1-\psi)v_t-\overline{\eta_v}-\eta_{vx}x_t]}\right] \\ = \;& logE_t\left[e^{(1-\gamma)\Delta y^F_{t+1}+(1+\omega+\psi-\gamma)\Delta x_{t+1}-\theta(1+\omega)(\pi^*-\pi_{t+1})+(\psi-\gamma)v_{t+1}+g_{t+1}}\right] - \frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)v_t - \overline{\eta_v} - \eta_{vx}x_t\right] \\ = \;& \Phi_t.\end{aligned}$$

Now, log linearized around $\Phi_t = \bar{\Phi}$:

$$\begin{aligned}
g_t &= log\left[1+\alpha\beta e^{\Phi_t}\right] \\
&\approx log\left[1+\alpha\beta e^{\bar{\Phi}}\right] + \frac{\alpha\beta e^{\bar{\Phi}}}{1+\alpha\beta e^{\bar{\Phi}}}(\Phi_t - \bar{\Phi}) \\
&= log\left[1+\alpha\beta e^{\bar{\Phi}}\right] - \frac{\alpha\beta e^{\bar{\Phi}}}{1+\alpha\beta e^{\bar{\Phi}}}\bar{\Phi} - \frac{\alpha\beta e^{\bar{\Phi}}}{1+\alpha\beta e^{\bar{\Phi}}}\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)v_t - \overline{\eta_v} - \eta_{vx}x_t\right] + \\
&\quad \frac{\alpha\beta e^{\bar{\Phi}}}{1+\alpha\beta e^{\bar{\Phi}}} logE_t\left[e^{(1-\gamma)\Delta y^F_{t+1}+(1+\omega+\psi-\gamma)\Delta x_{t+1}-\theta(1+\omega)(\pi^*-\pi_{t+1})+(\psi-\gamma)v_{t+1}+g_{t+1}}\right] \\
&= \underbrace{log\left[1+\alpha\beta e^{\bar{\Phi}}\right] - \frac{\alpha\beta e^{\bar{\Phi}}}{1+\alpha\beta e^{\bar{\Phi}}}\bar{\Phi} + \frac{\alpha\beta e^{\bar{\Phi}}}{1+\alpha\beta e^{\bar{\Phi}}}\frac{\psi-\gamma}{1-\gamma}\frac{\overline{\eta_v}}{\eta_{vv}}}_{\overline{\eta_g}} \underbrace{- \frac{\alpha\beta e^{\bar{\Phi}}}{1+\alpha\beta e^{\bar{\Phi}}}\frac{\psi-\gamma}{1-\gamma}\frac{1-\psi}{\eta_{vv}}}_{\eta_{gv}} v_t + \\
&\quad \underbrace{\frac{\alpha\beta e^{\bar{\Phi}}}{1+\alpha\beta e^{\bar{\Phi}}}\frac{\psi-\gamma}{1-\gamma}\frac{\eta_{vx}}{\eta_{vv}}}_{\eta_{gx}} x_t + \underbrace{\frac{\alpha\beta e^{\bar{\Phi}}}{1+\alpha\beta e^{\bar{\Phi}}}}_{\eta_{gg}} logE_t\left[e^{(1-\gamma)\Delta y^F_{t+1}+(1+\omega+\psi-\gamma)\Delta x_{t+1}-\theta(1+\omega)(\pi^*-\pi_{t+1})+(\psi-\gamma)v_{t+1}+g_{t+1}}\right] \\
&= \overline{\eta_g} + \eta_{gv}v_t + \eta_{gx}x_t + \eta_{gg} logE_t\left[e^{(1-\gamma)\Delta y^F_{t+1}+(1+\omega+\psi-\gamma)\Delta x_{t+1}-\theta(1+\omega)(\pi^*-\pi_{t+1})+(\psi-\gamma)v_{t+1}+g_{t+1}}\right].
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\bar{\Phi} &= -(1+\omega+\psi-\gamma)x_a\theta_y + \theta(1+\omega)\bar{\pi} + (\psi-\gamma)\bar{v} + \bar{g} + \\
&\quad \left[(1-\gamma) + (1+\omega+\psi-\gamma)x_a + \theta(1+\omega)\pi_a + (\psi-\gamma)v_a + g_a\right]\theta_y + \\
&\quad \frac{1}{2}\left[(1-\gamma) + (1+\omega+\psi-\gamma)x_a + \theta(1+\omega)\pi_a + (\psi-\gamma)v_a + g_a\right]^2\sigma_y^2 + \\
&\quad \frac{1}{2}\left[(1+\omega+\psi-\gamma)x_u + \theta(1+\omega)\pi_u + (\psi-\gamma)v_u + g_u\right]^2\sigma_u^2 - \\
&\quad \frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)\bar{v} - \overline{\eta_v} - \eta_{vx}\bar{x} + ((1-\psi)v_a - \eta_{vx}x_a)\theta_y\right]
\end{aligned}$$

To summarize, the log linearized system is:

$$\begin{aligned}
e^{-i_t} &= E_t[M^{\$}_{t,t+1}] && (35) \\
g_t &= \left(\frac{1+\theta\omega}{1-\theta}\right)\left[D_\pi + F_\pi(\pi_t - \overline{m_\pi})\right] + h_t - (\omega+\psi)x_t && (36) \\
v_t &= \frac{1}{(1-\psi)}\left\{\overline{\eta_v} + \eta_{vx}x_t + \eta_{vv} logE_t\left[e^{(1-\gamma)(v_{t+1}+\Delta y^F_{t+1}+\Delta x_{t+1})}\right]\right\} && (37) \\
h_t &= \overline{\eta_h} + \eta_{hv}v_t + \eta_{hx}x_t + \eta_{hh} logE_t\left[e^{(1-\gamma)(\Delta y^F_{t+1}+\Delta x_{t+1})+(1-\theta)(\pi^*-\pi_{t+1})+(\psi-\gamma)v_{t+1}+h_{t+1}}\right] && (38) \\
g_t &= \overline{\eta_g} + \eta_{gv}v_t + \eta_{gx}x_t + \eta_{gg} logE_t\left[e^{(1-\gamma)\Delta y^F_{t+1}+(1+\omega+\psi-\gamma)\Delta x_{t+1}-\theta(1+\omega)(\pi^*-\pi_{t+1})+(\psi-\gamma)v_{t+1}+g_{t+1}}\right] && (39) \\
i_t &= \bar{i} + i_x x_t + i_\pi(\pi_t - \pi^*) + u_t && (40) \\
u_{t+1} &= \phi_u u_t + \sigma_u \epsilon_{u,t+1} && (41) \\
\Delta y^F_{t+1} &= (1-\phi_a)\theta_y + \phi_a \Delta y^F_t + \sigma_y \epsilon_{a,t+1}, && (42)
\end{aligned}$$

where we have shut down the ARG processes from the original system.

D.3 Solution to the System

We guess the solutions to the system are affine in the state variables, thus:

$$\begin{aligned} x_t &= \bar{x} + x_a \Delta y_t^F + x_u u_t \\ \pi_t - \pi^* &= \bar{\pi} + \pi_a \Delta y_t^F + \pi_u u_t \\ h_t &= \bar{h} + h_a \Delta y_t^F + h_u u_t \\ g_t &= \bar{g} + g_a \Delta y_t^F + g_u u_t \\ v_t &= \bar{v} + v_a \Delta y_t^F + v_u u_t. \end{aligned}$$

From (36),

$$\begin{aligned} & \bar{g} + g_a \Delta y_t^F + g_u u_t \\ = & \left(\frac{1+\theta\omega}{1-\theta}\right) \left[D_\pi + F_\pi(\pi^* + \bar{\pi} + \pi_a \Delta y_t^F + \pi_u u_t - \overline{m_\pi})\right] + \bar{h} + h_a \Delta y_t^F + h_u u_t - (\omega+\psi)(\bar{x} + x_a \Delta y_t^F + x_u u_t), \end{aligned}$$

then use matching coefficients:

$$\begin{aligned} Const &: \left(\frac{1+\theta\omega}{1-\theta}\right) [D_\pi + F_\pi(\pi^* + \bar{\pi} - \overline{m_\pi})] + \bar{h} - (\omega+\psi)\bar{x} - \bar{g} = 0 \\ \Delta y_t^F &: \left(\frac{1+\theta\omega}{1-\theta}\right) F_\pi \pi_a + h_a - (\omega+\psi) x_a - g_a = 0 \\ u_t &: \left(\frac{1+\theta\omega}{1-\theta}\right) F_\pi \pi_u + h_u - (\omega+\psi) x_u - g_u = 0. \end{aligned}$$

From (38),

$$
\begin{aligned}
& \bar{h} + h_a \Delta y_t^F + h_u u_t \\
= \;& \overline{\eta_h} + \eta_{hv}(\bar{v} + v_a \Delta y_t^F + v_u u_t) + \eta_{hx}(\bar{x} + x_a \Delta y_t^F + x_u u_t) - (1-\gamma)\eta_{hh}(\bar{x} + x_a \Delta y_t^F + x_u u_t) + \\
& \eta_{hh} log E_t \Big[e^{(1-\gamma)(\Delta y_{t+1}^F + \bar{x} + x_a \Delta y_{t+1}^F + x_u u_{t+1}) - (1-\theta)(\bar{\pi} + \pi_a \Delta y_{t+1}^F + \pi_u u_{t+1})} \\
& \qquad\qquad\qquad\qquad e^{(\psi-\gamma)(\bar{v} + v_a \Delta y_{t+1}^F + v_u u_{t+1}) + \bar{h} + h_a \Delta y_{t+1}^F + h_u u_{t+1}} \Big] \\
= \;& \overline{\eta_h} + \eta_{hv}(\bar{v} + v_a \Delta y_t^F + v_u u_t) + \eta_{hx}(\bar{x} + x_a \Delta y_t^F + x_u u_t) - (1-\gamma)\eta_{hh}(\bar{x} + x_a \Delta y_t^F + x_u u_t) + \\
& \eta_{hh} \left[(1-\gamma)\bar{x} - (1-\theta)\bar{\pi} + (\psi-\gamma)\bar{v} + \bar{h} \right] + \eta_{hh} log E_t \Big[e^{[(1-\gamma)(1+x_a) - (1-\theta)\pi_a + (\psi-\gamma)v_a + h_a]\Delta y_{t+1}^F} \\
& \qquad\qquad\qquad\qquad e^{[(1-\gamma)x_u - (1-\theta)\pi_u + (\psi-\gamma)v_u + h_u]u_{t+1}} \Big] \\
= \;& \overline{\eta_h} + \eta_{hv}\bar{v} + \eta_{hx}\bar{x} + \eta_{hh}\left[-(1-\theta)\bar{\pi} + (\psi-\gamma)\bar{v} + \bar{h} \right] + \left[\eta_{hv} v_a + \eta_{hx} x_a - (1-\gamma)\eta_{hh} x_a \right] \Delta y_t^F \\
& \left[\eta_{hv} v_u + \eta_{hx} x_u - (1-\gamma)\eta_{hh} x_u \right] u_t + \eta_{hh} \times \\
& \Big\{ [(1-\gamma)(1+x_a) - (1-\theta)\pi_a + (\psi-\gamma)v_a + h_a] E_t[\Delta y_{t+1}^F] + \\
& \frac{1}{2}[(1-\gamma)(1+x_a) - (1-\theta)\pi_a + (\psi-\gamma)v_a + h_a]^2 var_t(\Delta y_{t+1}^F) + \\
& [(1-\gamma)x_u - (1-\theta)\pi_u + (\psi-\gamma)v_u + h_u] E_t[u_{t+1}] + \\
& \frac{1}{2}[(1-\gamma)x_u - (1-\theta)\pi_u + (\psi-\gamma)v_u + h_u]^2 var_t(u_{t+1}) \Big\} \\
= \;& \overline{\eta_h} + \eta_{hv}\bar{v} + \eta_{hx}\bar{x} + \eta_{hh}\left[-(1-\theta)\bar{\pi} + (\psi-\gamma)\bar{v} + \bar{h} \right] + \left[\eta_{hv} v_a + \eta_{hx} x_a - (1-\gamma)\eta_{hh} x_a \right] \Delta y_t^F \\
& \left[\eta_{hv} v_u + \eta_{hx} x_u - (1-\gamma)\eta_{hh} x_u \right] u_t + \eta_{hh} \times \\
& \Big\{ [(1-\gamma)(1+x_a) - (1-\theta)\pi_a + (\psi-\gamma)v_a + h_a][(1-\phi_a)\theta_y + \phi_a \Delta y_t^F] + \\
& \frac{1}{2}[(1-\gamma)(1+x_a) - (1-\theta)\pi_a + (\psi-\gamma)v_a + h_a]^2 \sigma_y^2 + \\
& [(1-\gamma)x_u - (1-\theta)\pi_u + (\psi-\gamma)v_u + h_u]\phi_u u_t + \\
& \frac{1}{2}[(1-\gamma)x_u - (1-\theta)\pi_u + (\psi-\gamma)v_u + h_u]^2 \sigma_u^2 \Big\},
\end{aligned}
$$

then use matching coefficients:

$$
\begin{aligned}
Const \;&:\; \bar{h} = \overline{\eta_h} + \eta_{hv}\bar{v} + \eta_{hx}\bar{x} + \eta_{hh}\left[-(1-\theta)\bar{\pi} + (\psi-\gamma)\bar{v} + \bar{h} \right] + \\
& \quad \eta_{hh} \{ [(1-\gamma)(1+x_a) - (1-\theta)\pi_a + (\psi-\gamma)v_a + h_a](1-\phi_a)\theta_y + \\
& \quad \frac{1}{2}[(1-\gamma)(1+x_a) - (1-\theta)\pi_a + (\psi-\gamma)v_a + h_a]^2 \sigma_y^2 + \frac{1}{2}[(1-\gamma)x_u - (1-\theta)\pi_u + (\psi-\gamma)v_u + h_u]^2 \sigma_u^2 \Big\} \\
\Delta y_t^F \;&:\; h_a = [\eta_{hv} v_a + \eta_{hx} x_a - (1-\gamma)\eta_{hh} x_a] + \eta_{hh}[(1-\gamma)(1+x_a) - (1-\theta)\pi_a + (\psi-\gamma)v_a + h_a]\phi_a \\
u_t \;&:\; h_u = [\eta_{hv} v_u + \eta_{hx} x_u - (1-\gamma)\eta_{hh} x_u] + \eta_{hh}[(1-\gamma)x_u - (1-\theta)\pi_u + (\psi-\gamma)v_u + h_u]\phi_u.
\end{aligned}
$$

From (39),

$$\begin{aligned}
& \bar{g} + g_a \Delta y_t^F + g_u u_t \\
= \quad & \overline{\eta_g} + \eta_{gv}(\bar{v} + v_a \Delta y_t^F + v_u u_t) + \eta_{gx}(\bar{x} + x_a \Delta y_t^F + x_u u_t) - (1+\omega+\psi-\gamma)\eta_{gg}(\bar{x} + x_a \Delta y_t^F + x_u u_t) + \\
& \eta_{gg} logE_t \Big[e^{(1-\gamma)\Delta y_{t+1}^F + (1+\omega+\psi-\gamma)(\bar{x} + x_a \Delta y_{t+1}^F + x_u u_{t+1}) + \theta(1+\omega)(\bar{\pi} + \pi_a \Delta y_{t+1}^F + \pi_u u_{t+1})} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad e^{(\psi-\gamma)(\bar{v} + v_a \Delta y_{t+1}^F + v_u u_{t+1}) + \bar{g} + g_a \Delta y_{t+1}^F + g_u u_{t+1}} \Big] \\
= \quad & \overline{\eta_g} + \eta_{gv}(\bar{v} + v_a \Delta y_t^F + v_u u_t) + \eta_{gx}(\bar{x} + x_a \Delta y_t^F + x_u u_t) - (1+\omega+\psi-\gamma)\eta_{gg}(\bar{x} + x_a \Delta y_t^F + x_u u_t) + \\
& \eta_{gg} \left[(1+\omega+\psi-\gamma)\bar{x} + \theta(1+\omega)\bar{\pi} + (\psi-\gamma)\bar{v} + \bar{g}\right] + \eta_{gg} logE_t \Big[e^{[(1-\gamma) + (1+\omega+\psi-\gamma)x_a + \theta(1+\omega)\pi_a + (\psi-\gamma)v_a + g_a]\Delta y_{t+1}^F} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad e^{[(1+\omega+\psi-\gamma)x_u + \theta(1+\omega)\pi_u + (\psi-\gamma)v_u + g_u]u_{t+1}} \Big] \\
= \quad & \overline{\eta_g} + \eta_{gv}\bar{v} + \eta_{gx}\bar{x} + \eta_{gg}\left[\theta(1+\omega)\bar{\pi} + (\psi-\gamma)\bar{v} + \bar{g}\right] + \left[\eta_{gv} v_a + \eta_{gx} x_a - (1+\omega+\psi-\gamma)\eta_{gg} x_a\right] \Delta y_t^F \\
& \left[\eta_{gv} v_u + \eta_{gx} x_u - (1+\omega+\psi-\gamma)\eta_{gg} x_u\right] u_t + \eta_{gg} \times \\
& \Big\{ [(1-\gamma) + (1+\omega+\psi-\gamma)x_a + \theta(1+\omega)\pi_a + (\psi-\gamma)v_a + g_a] E_t[\Delta y_{t+1}^F] + \\
& \frac{1}{2}[(1-\gamma) + (1+\omega+\psi-\gamma)x_a + \theta(1+\omega)\pi_a + (\psi-\gamma)v_a + g_a]^2 var_t(\Delta y_{t+1}^F) + \\
& [(1+\omega+\psi-\gamma)x_u + \theta(1+\omega)\pi_u + (\psi-\gamma)v_u + g_u] E_t[u_{t+1}] + \\
& \frac{1}{2}[(1+\omega+\psi-\gamma)x_u + \theta(1+\omega)\pi_u + (\psi-\gamma)v_u + g_u]^2 var_t(u_{t+1}) \Big\} \\
= \quad & \overline{\eta_g} + \eta_{gv}\bar{v} + \eta_{gx}\bar{x} + \eta_{gg}\left[\theta(1+\omega)\bar{\pi} + (\psi-\gamma)\bar{v} + \bar{g}\right] + \left[\eta_{gv} v_a + \eta_{gx} x_a - (1+\omega+\psi-\gamma)\eta_{gg} x_a\right] \Delta y_t^F \\
& \left[\eta_{gv} v_u + \eta_{gx} x_u - (1+\omega+\psi-\gamma)\eta_{gg} x_u\right] u_t + \eta_{gg} \times \\
& \Big\{ [(1-\gamma) + (1+\omega+\psi-\gamma)x_a + \theta(1+\omega)\pi_a + (\psi-\gamma)v_a + g_a][(1-\phi_a)\theta_y + \phi_a \Delta y_t^F] + \\
& \frac{1}{2}[(1-\gamma) + (1+\omega+\psi-\gamma)x_a + \theta(1+\omega)\pi_a + (\psi-\gamma)v_a + g_a]^2 \sigma_y^2 + \\
& [(1+\omega+\psi-\gamma)x_u + \theta(1+\omega)\pi_u + (\psi-\gamma)v_u + g_u]\phi_u u_t + \\
& \frac{1}{2}[(1+\omega+\psi-\gamma)x_u + \theta(1+\omega)\pi_u + (\psi-\gamma)v_u + g_u]^2 \sigma_u^2 \Big\},
\end{aligned}$$

then use matching coefficients:

$$\begin{aligned}
Const \quad : \quad & \bar{g} = \overline{\eta_g} + \eta_{gv}\bar{v} + \eta_{gx}\bar{x} + \eta_{gg}\left[\theta(1+\omega)\bar{\pi} + (\psi-\gamma)\bar{v} + \bar{g}\right] + \\
& \eta_{gg} \{ [(1-\gamma) + (1+\omega+\psi-\gamma)x_a + \theta(1+\omega)\pi_a + (\psi-\gamma)v_a + g_a](1-\phi_a)\theta_y + \\
& \frac{1}{2}[(1-\gamma) + (1+\omega+\psi-\gamma)x_a + \theta(1+\omega)\pi_a + (\psi-\gamma)v_a + g_a]^2 \sigma_y^2 + \\
& \frac{1}{2}[(1+\omega+\psi-\gamma)x_u + \theta(1+\omega)\pi_u + (\psi-\gamma)v_u + g_u]^2 \sigma_u^2 \Big\} \\
\Delta y_t^F \quad : \quad & g_a = [\eta_{gv} v_a + \eta_{gx} x_a - (1+\omega+\psi-\gamma)\eta_{gg} x_a] + \\
& \eta_{gg}[(1-\gamma) + (1+\omega+\psi-\gamma)x_a + \theta(1+\omega)\pi_a + (\psi-\gamma)v_a + g_a]\phi_a \\
u_t \quad : \quad & g_u = [\eta_{gv} v_u + \eta_{gx} x_u - (1+\omega+\psi-\gamma)\eta_{gg} x_u] + \eta_{gg}[(1+\omega+\psi-\gamma)x_u + \theta(1+\omega)\pi_u + (\psi-\gamma)v_u + g_u]\phi_u.
\end{aligned}$$

Then, combining (35) and (40),

$$
\begin{aligned}
& e^{-\bar{i}-i_x x_t-i_\pi(\pi_t-\pi^*)-u_t} \\
= \quad & E_t\left[\frac{\beta}{\Pi^*}\left(\frac{Y_{t+1}^F}{Y_t^F}\right)^{-\psi}\left(\frac{X_{t+1}}{X_t}\right)^{-\psi}\left(\frac{\Pi^*}{\Pi_{t+1}}\right)\left(\frac{V_{t+1}}{E_t\left[V_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\psi-\gamma}\right] \\
= \quad & E_t\left[\frac{\beta}{\Pi^*}\left(\frac{Y_{t+1}^F}{Y_t^F}\right)^{-\psi}\left(\frac{X_{t+1}}{X_t}\right)^{-\psi}\left(\frac{\Pi^*}{\Pi_{t+1}}\right)\frac{e^{(\psi-\gamma)(v_{t+1}+\Delta y_{t+1}^F+\Delta x_{t+1})}}{e^{\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}[(1-\psi)v_t-\overline{\eta_v}-\eta_{vx}x_t]}}\right] \\
= \quad & E_t\left[e^{log\beta-\pi^*-\psi\Delta y_{t+1}^F-\psi\Delta x_{t+1}+(\pi^*-\pi_{t+1})}\frac{e^{(\psi-\gamma)(v_{t+1}+\Delta y_{t+1}^F+\Delta x_{t+1})}}{e^{\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}[(1-\psi)v_t-\overline{\eta_v}-\eta_{vx}x_t]}}\right].
\end{aligned}
$$

Taking log on both sides,

$$
\begin{aligned}
& -\bar{i}-i_x x_t-i_\pi(\pi_t-\pi^*)-u_t \\
= \quad & log\beta-\pi^*-\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)v_t-\overline{\eta_v}-\eta_{vx}x_t\right]+\gamma x_t+logE_t\left[e^{-\gamma\Delta y_{t+1}^F-\gamma x_{t+1}+(\psi-\gamma)v_{t+1}+(\pi^*-\pi_{t+1})}\right] \\
= \quad & log\beta-\pi^*-\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)v_t-\overline{\eta_v}-\eta_{vx}x_t\right]+\gamma x_t+ \\
& logE_t\left[e^{-\gamma\Delta y_{t+1}^F-\gamma(\bar{x}+x_a\Delta y_{t+1}^F+x_u u_{t+1})+(\psi-\gamma)(\bar{v}+v_a\Delta y_{t+1}^F+v_u u_{t+1})-(\bar{\pi}+\pi_a\Delta y_{t+1}^F+\pi_u u_{t+1})}\right] \\
= \quad & log\beta-\pi^*-\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)v_t-\overline{\eta_v}-\eta_{vx}x_t\right]+\gamma x_t-\gamma\bar{x}+(\psi-\gamma)\bar{v}-\bar{\pi} \\
& logE_t\left[e^{[-\gamma(1+x_a)+(\psi-\gamma)v_a-\pi_a]\Delta y_{t+1}^F+[-\gamma x_u+(\psi-\gamma)v_u-\pi_u]u_{t+1}}\right] \\
= \quad & log\beta-\pi^*-\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)(\bar{v}+v_a\Delta y_t^F+v_u u_t)-\overline{\eta_v}-\eta_{vx}(\bar{x}+x_a\Delta y_t^F+x_u u_t)\right]+\gamma(\bar{x}+x_a\Delta y_t^F+x_u u_t)- \\
& \gamma\bar{x}+(\psi-\gamma)\bar{v}-\bar{\pi}+\left[-\gamma(1+x_a)+(\psi-\gamma)v_a-\pi_a\right]\left[(1-\phi_a)\theta_y+\phi_a\Delta y_t^F\right]+\frac{1}{2}\left[-\gamma(1+x_a)+(\psi-\gamma)v_a-\pi_a\right]^2\sigma_y^2+ \\
& \left[-\gamma x_u+(\psi-\gamma)v_u-\pi_u\right]\phi_u u_t+\frac{1}{2}\left[-\gamma x_u+(\psi-\gamma)v_u-\pi_u\right]^2\sigma_u^2,
\end{aligned}
$$

then use matching coefficients:

$$
\begin{aligned}
Const \quad : \quad & -\bar{i}-i_x\bar{x}-i_\pi\bar{\pi}=log\beta-\pi^*-\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)\bar{v}-\overline{\eta_v}-\eta_{vx}\bar{x}\right]+(\psi-\gamma)\bar{v}-\bar{\pi}+ \\
& \left[-\gamma(1+x_a)+(\psi-\gamma)v_a-\pi_a\right](1-\phi_a)\theta_y+\frac{1}{2}\left[-\gamma(1+x_a)+(\psi-\gamma)v_a-\pi_a\right]^2\sigma_y^2+ \\
& \frac{1}{2}\left[-\gamma x_u+(\psi-\gamma)v_u-\pi_u\right]^2\sigma_u^2 \\
\Delta y_t^F \quad : \quad & -i_x x_a-i_\pi\pi_a=-\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)v_a-\eta_{vx}x_a\right]+\gamma x_a+\left[-\gamma(1+x_a)+(\psi-\gamma)v_a-\pi_a\right]\phi_a \\
u_t \quad : \quad & -i_x x_u-i_\pi\pi_u-1=-\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)v_u-\eta_{vx}x_u\right]+\gamma x_u+\left[-\gamma x_u+(\psi-\gamma)v_u-\pi_u\right]\phi_u.
\end{aligned}
$$

Finally, take (37),

$$
\begin{aligned}
& \bar{v} + v_a \Delta y_t^F + v_u u_t \\
= \; & \frac{1}{(1-\psi)} \left\{ \overline{\eta_v} + \eta_{vx}(\bar{x} + x_a \Delta y_t^F + x_u u_t) - \eta_{vv}(1-\gamma)(\bar{x} + x_a \Delta y_t^F + x_u u_t) + \right. \\
& \left. \eta_{vv} logE_t \left[e^{(1-\gamma)\left[(\bar{v}+v_a\Delta y_{t+1}^F + v_u u_{t+1}) + \Delta y_{t+1}^F + (\bar{x} + x_a \Delta y_{t+1}^F + x_u u_{t+1})\right]} \right] \right\} \\
= \; & \frac{1}{(1-\psi)} \left\{ \overline{\eta_v} + \eta_{vx}(\bar{x} + x_a \Delta y_t^F + x_u u_t) - \eta_{vv}(1-\gamma)(\bar{x} + x_a \Delta y_t^F + x_u u_t) + \right. \\
& \left. \eta_{vv}(1-\gamma)(\bar{v} + \bar{x}) + \eta_{vv} logE_t \left[e^{(1-\gamma)\left[(1+v_a+x_a)\Delta y_{t+1}^F + (v_u + x_u) u_{t+1}\right]} \right] \right\} \\
= \; & \frac{1}{(1-\psi)} \left\{ \overline{\eta_v} + \eta_{vx}(\bar{x} + x_a \Delta y_t^F + x_u u_t) - \eta_{vv}(1-\gamma)(\bar{x} + x_a \Delta y_t^F + x_u u_t) + \right. \\
& \eta_{vv}(1-\gamma)(\bar{v} + \bar{x}) + \eta_{vv}(1-\gamma)(1+v_a+x_a)\left[(1-\phi_a)\theta_y + \phi_a \Delta y_t^F\right] + \frac{1}{2}\eta_{vv}(1-\gamma)^2(1+v_a+x_a)^2\sigma_y^2 + \\
& \left. \eta_{vv}(1-\gamma)(v_u + x_u)\phi_u u_t + \frac{1}{2}\eta_{vv}(1-\gamma)^2(v_u+x_u)^2\sigma_u^2 \right\},
\end{aligned}
$$

then use matching coefficients:

$$
\begin{aligned}
Const \quad : \quad & (1-\psi)\bar{v} = \overline{\eta_v} + \eta_{vx}\bar{x} - \eta_{vv}(1-\gamma)\bar{x} + \eta_{vv}(1-\gamma)(\bar{v}+\bar{x}) + \eta_{vv}(1-\gamma)(1+v_a+x_a)(1-\phi_a)\theta_y + \\
& \frac{1}{2}\eta_{vv}(1-\gamma)^2(1+v_a+x_a)^2\sigma_y^2 + \frac{1}{2}\eta_{vv}(1-\gamma)^2(v_u+x_u)^2\sigma_u^2 \\
\Delta y_t^F \quad : \quad & (1-\psi)v_a = \eta_{vx}x_a - \eta_{vv}(1-\gamma)x_a + \eta_{vv}(1-\gamma)(1+v_a+x_a)\phi_a \\
u_t \quad : \quad & (1-\psi)v_u = \eta_{vx}x_u - \eta_{vv}(1-\gamma)x_u + \eta_{vv}(1-\gamma)(v_u+x_u)\phi_u.
\end{aligned}
$$

D.4 Pricing Kernel in Sticky Price Economy

The real pricing kernel in this economy is:

$$
\begin{aligned}
M_{t,t+1} &= \beta \left(\frac{Y_{t+1}^F}{Y_t^F}\right)^{-\psi} \left(\frac{X_{t+1}}{X_t}\right)^{-\psi} \left(\frac{V_{t+1}}{E_t\left[V_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}} \right)^{\psi-\gamma} \\
&= \beta \left(\frac{Y_{t+1}^F}{Y_t^F}\right)^{-\psi} \left(\frac{X_{t+1}}{X_t}\right)^{-\psi} \left(\frac{e^{(\psi-\gamma)(v_{t+1}+\Delta y_{t+1}^F + \Delta x_{t+1})}}{e^{\frac{\psi-\gamma}{1-\gamma} logE_t\left[e^{(1-\gamma)(v_{t+1}+\Delta y_{t+1}^F+\Delta x_{t+1})}\right]}} \right) \\
&= \beta \left(\frac{Y_{t+1}^F}{Y_t^F}\right)^{-\gamma} \left(\frac{X_{t+1}}{X_t}\right)^{-\gamma} \left(\frac{e^{(\psi-\gamma)v_{t+1}}}{e^{\frac{\psi-\gamma}{1-\gamma}\frac{1}{\eta_{vv}}\left[(1-\psi)v_t - \overline{\eta_v} - \eta_{vx}x_t\right]}} \right),
\end{aligned}
$$

and the nominal pricing kernel is:

$$
M_{t,t+1}^{\$} = \frac{M_{t,t+1}}{\Pi^*}\frac{\Pi^*}{\Pi_{t+1}}.
$$

Let $M^*_{t,t+1} = \begin{cases} M_{t,t+1} & \text{if } I^\$ = 0; \\ M^\$_{t,t+1} & \text{if } I^\$ = 1. \end{cases}$, then

$$\begin{aligned}
-logM^*_{t,t+1} &= -log\beta + \gamma\Delta y^F_{t+1} + \gamma\Delta x_{t+1} - (\psi-\gamma)v_{t+1} + \left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{1}{\eta_{vv}}\left[(1-\psi)v_t - \overline{\eta_v} - \eta_{vx}x_t\right] + \\
&\quad I^\$\left[(\pi_{t+1}-\pi^*)+\pi^*\right] \\
&= -log\beta - \left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{1}{\eta_{vv}}\overline{\eta_v} + I^\$\pi^* - \left[\left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{\eta_{vx}}{\eta_{vv}}+\gamma\right]x_t + \left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{1-\psi}{\eta_{vv}}v_t + \\
&\quad \gamma\bar{x} - (\psi-\gamma)\bar{v} + I^\$\bar{\pi} + \left[\gamma(1+x_a) - (\psi-\gamma)v_a + I^\$\pi_a\right]\Delta y^F_{t+1} + \left[\gamma x_u - (\psi-\gamma)v_u + I^\$\pi_u\right]u_{t+1} \\
&= -log\beta - \left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{1}{\eta_{vv}}\overline{\eta_v} + I^\$\pi^* - \left[\left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{\eta_{vx}}{\eta_{vv}}+\gamma\right](\bar{x} + x_a\Delta y^F_t + x_u u_t) + \\
&\quad \left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{1-\psi}{\eta_{vv}}(\bar{v} + v_a\Delta y^F_t + v_u u_t) + \gamma\bar{x} - (\psi-\gamma)\bar{v} + I^\$\bar{\pi} + \\
&\quad \left[\gamma(1+x_a) - (\psi-\gamma)v_a + I^\$\pi_a\right]\left[(1-\phi_a)\theta_y + \phi_a\Delta y^F_t + \sigma_y\epsilon_{a,t+1}\right] + \\
&\quad \left[\gamma x_u - (\psi-\gamma)v_u + I^\$\pi_u\right]\left[\phi_u u_t + \sigma_u\epsilon_{u,t+1}\right] \\
&= -log\beta - \left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{1}{\eta_{vv}}\overline{\eta_v} + I^\$\pi^* - \left[\left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{\eta_{vx}}{\eta_{vv}}\right]\bar{x} + \left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{1-\psi}{\eta_{vv}}\bar{v} - (\psi-\gamma)\bar{v} + \\
&\quad I^\$\bar{\pi} + \left[\gamma(1+x_a) - (\psi-\gamma)v_a + I^\$\pi_a\right](1-\phi_a)\theta_y + \\
&\quad \left\{-\left[\left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{\eta_{vx}}{\eta_{vv}}+\gamma\right]x_a + \left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{1-\psi}{\eta_{vv}}v_a + \left[\gamma(1+x_a) - (\psi-\gamma)v_a + I^\$\pi_a\right]\phi_a\right\}\Delta y^F_t + \\
&\quad \left\{-\left[\left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{\eta_{vx}}{\eta_{vv}}+\gamma\right]x_u + \left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{1-\psi}{\eta_{vv}}v_u + \left[\gamma x_u - (\psi-\gamma)v_u + I^\$\pi_u\right]\phi_u\right\}u_t + \\
&\quad \left[\gamma(1+x_a) - (\psi-\gamma)v_a + I^\$\pi_a\right]\sigma_y\epsilon_{a,t+1} + \left[\gamma x_u - (\psi-\gamma)v_u + I^\$\pi_u\right]\sigma_u\epsilon_{u,t+1} \\
&= \Gamma_0 + \Gamma_a\Delta y^F_t + \Gamma_u u_t + \lambda_a\sigma_y\epsilon_{a,t+1} + \lambda_u\sigma_u\epsilon_{u,t+1},
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_0 &= -log\beta - \left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{1}{\eta_{vv}}\overline{\eta_v} + I^\$(\pi^* + \bar{\pi}) - \left[\left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{\eta_{vx}}{\eta_{vv}}\right]\bar{x} + \left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{1-\psi}{\eta_{vv}}\bar{v} - (\psi-\gamma)\bar{v} + \\
&\quad \left[\gamma(1+x_a) - (\psi-\gamma)v_a + I^\$\pi_a\right](1-\phi_a)\theta_y \\
\Gamma_a &= -\left[\left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{\eta_{vx}}{\eta_{vv}}+\gamma\right]x_a + \left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{1-\psi}{\eta_{vv}}v_a + \left[\gamma(1+x_a) - (\psi-\gamma)v_a + I^\$\pi_a\right]\phi_a \\
\Gamma_u &= -\left[\left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{\eta_{vx}}{\eta_{vv}}+\gamma\right]x_u + \left(\frac{\psi-\gamma}{1-\gamma}\right)\frac{1-\psi}{\eta_{vv}}v_u + \left[\gamma x_u - (\psi-\gamma)v_u + I^\$\pi_u\right]\phi_u \\
\lambda_a &= \left[\gamma(1+x_a) - (\psi-\gamma)v_a + I^\$\pi_a\right] \\
\lambda_u &= \left[\gamma x_u - (\psi-\gamma)v_u + I^\$\pi_u\right].
\end{aligned}$$

D.5 The Term Structure of Interest Rate

$$\begin{aligned}
e^{-ny_t^{(n)}} &= E_t\left[e^{m_{t,t+1}-(n-1)y_{t+1}^{(n-1)}}\right] \\
&= E_t\left[e^{-\Gamma_0-\Gamma_a\Delta y_t^F-\Gamma_u u_t-\lambda_a\sigma_y\epsilon_{a,t+1}-\lambda_u\sigma_u\epsilon_{u,t+1}}e^{-A_{n-1}-B_{a,n-1}\Delta y_{t+1}^F-B_{u,n-1}u_{t+1}}\right] \\
&= e^{-\Gamma_0-\Gamma_a\Delta y_t^F-\Gamma_u u_t-A_{n-1}}E_t\left[e^{-\lambda_a\sigma_y\epsilon_{a,t+1}-B_{a,n-1}\Delta y_{t+1}^F}\right]E_t\left[e^{-\lambda_u\sigma_u\epsilon_{u,t+1}-B_{u,n-1}u_{t+1}}\right] \\
&= e^{\{-\Gamma_0-\Gamma_a\Delta y_t^F-\Gamma_u u_t-A_{n-1}-B_{a,n-1}[(1-\phi_a)\theta_y+\phi_a\Delta y_t]+\frac{1}{2}(\lambda_a+B_{a,n-1})^2\sigma_y^2\}}\times \\
&\quad e^{\{-B_{u,n-1}\phi_u u_t+\frac{1}{2}(\lambda_u+B_{u,n-1})^2\sigma_u^2\}}
\end{aligned}$$

Since $e^{-ny_t^{(n)}}=e^{-A_n-B_{a,n}\Delta y_t^F-B_{u,n}u_t}$, we can match coefficients to find A_n, $B_{a,n}$ and $B_{u,n}$:

$$\begin{aligned}
-A_n &= -\Gamma_0-A_{n-1}-B_{a,n-1}(1-\phi_a)\theta_y+\frac{1}{2}(\lambda_a+B_{a,n-1})^2\sigma_y^2+\frac{1}{2}(\lambda_u+B_{u,n-1})^2\sigma_u^2 \\
-B_{a,n} &= -\Gamma_a-B_{a,n-1}\phi_a \\
-B_{u,n} &= -\Gamma_u-B_{u,n-1}\phi_u
\end{aligned}$$

where $A_0=B_{a,0}=B_{u,0}=0$.