The Hazards of Debt: Rollover Freezes, Incentives, and Bailouts

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Abstract

We investigate the trade-off between incentive provision and rollover freezes for a firm that finances the holding of a long-term asset using staggered short-term debt contracts, and which has a manager who can risk-shift between a high-mean, low-volatility good asset and a low-mean, high-volatility bad asset at any point in time. First, we find that there is such a thing as debt that is too short-term: the optimal maturity is just short enough to prevent the manager from risk-shifting when the firm is in good health, and is an interior solution for a wide variety of parameters. Shorter maturities inefficiently worsen rollover risk. Second, we find that allowing the manager to risk-shift during a freeze may actually increase creditor confidence. Debt policy should therefore not prevent the manager from holding what may appear to be otherwise low-mean assets that have option value during a freeze. Third, we highlight that a limited but not perfectly reliable form of emergency financing during a freeze - a “bailout" - may improve the terms of the trade-off and increase total ex-ante value by instilling confidence in the creditor markets. This holds even when including incentive effects and government losses since expected losses are non-monotonic in bailout reliability.

Keywords: Rollover risk; rollover freezes; risk-shifting; optimal maturity; bailouts; financial firms; liquidity

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1 Introduction

Is the use of short-term debt optimal? Recent research has focused on the role of a freeze in short-term debt markets as a leading amplification mechanism that led to the worst financial crisis since the Great Depression.\(^1\) The basic premise is that the non-bank financial sector, which experienced rapid growth in the early and mid-2000’s, and which relied heavily on staggered short-term debt to finance risky long-term and illiquid assets, experienced a *rollover freeze* during the crisis. Short-term creditors refused to roll over their debt for fear of future deteriorations in the real estate market, leading to financial distress for those firms far exceeding the level of losses (Brunnermeier (2009)). More formally, He and Xiong (2009a) analyze this "rollover risk" and point out that financing an illiquid long-term asset with staggered short-term debt leads to a dynamic coordination problem among creditors and possibly inefficient liquidations.

An alternative literature, dating back to Calomiris and Kahn (1991), emphasizes the role of short-term debt as a disciplining device for moral hazard, for example to prevent risk-shifting by managers (Jensen and Meckling (1976)). Kashyap, Rajan, and Stein (2008) note that "short-term debt may reflect a privately optimal response to governance problems." Under this premise, short-term debt for the non-bank sector is value-increasing much in the way depositors are value-increasing for banking sector: the fragility of the institution itself provides incentives for depositors to monitor management and thus mitigates agency issues (Diamond and Rajan (2001), Diamond and Rajan (2000)). Nevertheless, the literature has yet to fully resolve the trade-off between incentives and rollover risk, the latter of which was, if not the match that lit the fire, arguably the accelerant that set the crisis ablaze.

In this paper, we attempt to reconcile these views. We consider a model of a firm subject to rollover freezes (He and Xiong (2009a)) and introduce both equity and an equity-holding manager who can risk-shift between a high-mean, low-volatility asset and a low-mean, high-volatility asset, but whose asset choice cannot be contracted upon. Our first contribution is to show that, even thought short-term induces the manager to hold the high-mean asset more often, debt maturity can indeed be too short-term: excessively short maturities reduce ex ante firm value through the possibility of rollover freezes and inefficient liquidation. The optimal maturity, which trades off incentives and rollover risk, rests at an interior solution for a wide variety of parameterizations.

Second, we show that risk-shifting in a dynamic context is not always value-decreasing.

\(^1\)See, for example, He and Xiong (2009a), Morris and Shin (2009), Acharya (2009), and Brunnermeier (2009).
Risk-shifting during a run actually increases firm value, even though, importantly, the low-mean asset is dominated by the high-mean asset in expected value. Risk-shifting increases the incentives of future creditors, whose debt has yet to mature, to roll over their debt (i.e., it reduces their incentive to ‘run’), and thus reduces the ex ante severity of rollover freezes. Counterintuitively, allowing a manager to risk-shift and take on an asset during a rollover freeze that is otherwise dominated in expected value may actually improve creditor confidence. Our point here is not to say that managers should seek wildly bad assets during a freeze, but rather to highlight that debt policy (and firm policies in general) should not be so stringent as to inadvertently prevent managers from holding assets or taking actions that would otherwise be deemed too risky or otherwise bad ideas during normal times, as desperate times may call for desperate measures.

Third, our model allows us to compute the ex ante cost and benefits of an outsider stepping in to provide emergency financing during a rollover freeze, which we broadly term a “bailout” by the government. We find that even when considering worsened incentives and including expected losses to the government, a limited bailout policy can increase total value (debt plus equity less bailout losses) by improving the terms of the tradeoff between incentives and rollover risk. The primary benefit of such a policy comes not through directly saving the firm from liquidation, but indirectly by establishing creditor confidence and reducing ex ante rollover risk. Importantly, in contrast to existing literature which analyzes the impact of a bailout policy through its effect on ruling out so-called run equilibria, our model allows us to parameterize the reliability of a bailout within an equilibrium that allows for the occurrence of freezes. This allows us to compute the ex ante bailout cost accounting for its impact on incentives and runs.

Although several caveats attach to our conclusions, which we discuss shortly, the question of how short-term is too short-term with respect to debt maturity and how government intervention in these markets affects incentives is of vital importance in light of the billions of taxpayer dollars committed by governments around the world to save many large financial institutions from freezes in short-term debt markets. These bailouts have arguably staved off the worst-case outcome of a complete collapse of the financial system and the possibility of a second Great Depression. However, as we emerge from this crisis, news of Wall Street pay returning to record levels have raised fresh questions about whether government bailouts have irreparably weakened future incentives. Our model is stylized, yet attempts to address these questions in an integrated fashion.

In more detail, our point of departure is the dynamic rollover risk model of He and Xiong (2009a). A financial firm finances holdings of long-term illiquid assets using staggered short-
term debt. The fundamentals of the asset evolve stochastically through time. The staggered debt maturity structure, along with a significant maturity mismatch between assets and liabilities, creates a dynamic coordination problem, or low “creditor confidence,” where creditors refuse to roll over their debt when firm fundamentals are sufficiently low. Importantly, these rollover freezes, or “dynamic debt runs,” are unlike static bank runs in that the time-varying firm fundamentals may improve before many other creditors have refused to roll over their financing. Thus a maturing creditor’s decision to roll over his debt depends on his anticipation of whether future creditors will roll over their debt, his estimate of what firm fundamentals will be at that future point in time, and how much emergency financing will be provided to the firm in the interim. Failing to survive a freeze results in distressed liquidation, so rollover freezes reduce firm value ex ante. These freezes are worse for shorter maturities: even though each maturing creditor can expect their debt to come due more quickly, they also expect other debtors to come due more quickly as well, exacerbating the coordination problem and leading to more distressed liquidations.

To this benchmark model, we introduce equity and thus present a complete balance sheet of the firm. We introduce a manager who is compensated via equity alone and determines whether the firm holds a high-mean, low volatility asset (the good asset), or a high-volatility, low-mean asset (the bad asset). At each point in time, the manager can switch between the good asset and bad asset, and his choice cannot be contracted upon. On the equity side, we now have a classic risk-shifting problem where the manager has an incentive to hold the bad asset when firm fundamentals are low, diluting the value of debt and lowering overall firm value.

In equilibrium, risk-shifting interacts with creditor confidence and rollover freezes in two ways. First, risk-shifting outside of a run situation (which we term preemptive risk-shifting) is an inefficient transfer of value from creditors to equity. However, our analysis highlights a second channel where risk-shifting and holding the high-volatility asset during a freeze often increases firm value due to the dynamic nature of a rollover freeze. The intuition is two-fold. First, increasing the volatility of the firm fundamental increases the “speed” at which it moves and thus the likelihood that the firm escapes the freeze, a direct effect. Second, this direct effect feeds through and results indirectly in improved creditor confidence, since in equilibrium, creditors anticipate that managers will hold the higher volatility asset during

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2 Our model can apply to non-financial firms as well. Indeed, many firms in the Asian crisis pursued many real investment opportunities financed with short-term debt. The subsequent squeeze in credit markets led many firms to abandon projects their real investment projects for little value.

3 We use the terms “rollover freezes” and “dynamic debt runs” interchangeably, although we generally stick to the “freeze” terminology rather than “run” to highlight the difference with static bank runs.
a freeze, which alleviates their concern about what other creditors will do in the near future.

This analysis underscores the intuition behind our first two results. The first result was that debt can be excessively short-term by leading to excessive freezes, even when accounting for the possibility of risk-shifting. Shortening the maturity of debt has three effects. First, it has a positive effect on value by disciplining preemptive risk-shifting. Notably, long-maturity debt that completely matches the maturity of assets is undesirable due to excessive preemptive risk-shifting. Second, shortening the maturity structure has a negative effect on value since the firm is less likely to survive a given freeze, due to the direct effect that there is very little time for fundamentals to recover before the firm is liquidated from lack of funding. This effect turns out to be small relative to the third effect, where shortening the maturity reduces creditor confidence and leads to more freezes and much lower value ex ante. The optimal maturity rests at an interior solution for a wide variety of parameterizations, and provides just enough discipline to eliminate preemptive risk-shifting.

The intuition behind our second result, that debt policy should not fully prevent the manager from holding the bad asset particularly during a freeze, stems from the fact that the interests of future, non-maturing creditors and current, maturing creditors diverge during a freeze. Effectively, non-maturing creditors are junior to maturing creditors and thus have more convex interests in that they want the firm to recover quickly (or at least survive long enough until they themselves mature) to avoid being saddled with inefficient liquidation. Consequently, they might actually prefer more volatility and a higher option value to the firm during a freeze as it can increase the chance to recover. In equilibrium, when all agents anticipate that managers will hold the high-volatility asset during the run, a maturing creditor will be less worried about future creditors’ motives to withdraw funding, which means other creditors will be less worried about other creditors withdrawing funding, and so forth. This means allowing the bad manager to hold the bad asset during a freeze results in fewer freezes and improved value, relative to the case where the manager is restricted to only ever hold the good asset.

The next part of our analysis considers how moral hazard and freezes vary with policies that provide emergency financing to the firm during a freeze by incentivizing debtors to roll over their debt, which we broadly term as a bailout. Our thought experiment is to ask whether total value is worsened by such a policy in a stylized setting where we think of our firm as representing the broad financial sector. In our model, an external entity, for example the government, provides creditors just enough money to rollover their debt on the margin during such a freeze. However, it only does so probabilistically in the sense that this emergency financing is not limitless and may disappear at some point, in which case the sector
experiences a severe liquidation cost, e.g., large costs associated with systemic failure. We parameterize the reliability of such emergency financing and compute the optimal reliability considering the endogenous effect of emergency financing on rollover freezes, moral hazard, expected government losses, and total value (ex ante value of debt plus equity less losses).

We find a non-zero optimal reliability improves total ex ante value. Increasing the reliability of bailouts affects total value through three effects. First, there is a positive direct effect: a more reliable bailout saves the sector from inefficient liquidation more often. This effect turns out to be relatively small. Second, there is a positive indirect effect on creditor confidence: a longer government lifeline means that a maturing creditor’s fear of whether future creditors will roll over is alleviated, feeding back in equilibrium into increased confidence and decreased likelihood of freezes ex ante. This effect is value-increasing and can be quite large. Third, there is a negative incentive effect: more reliable bailouts encourage the manager to hold the bad asset outside of a run situation more often. This is in and of itself bad for total value since the bad asset has a low mean. However, it is much worse, as it feeds back into high government losses. This is because with a high bailout reliability, rescuing the sector will only result in managers again holding the bad asset even after a freeze has been escaped. Creditors will then require a large amount to incentivize them to roll over their debt during a freeze, as the firm is fundamentally a bad asset.

Expected government losses are highly non-monotonic in bailout reliability due to these feedback effects. The optimal reliability of emergency financing is mild and just reliable enough to avoid the manager holding the bad asset outside of a freeze, but is non-zero in that it provides money to creditors during a freeze in order to boost creditor confidence. By avoiding preemptive risk-shifting, a mild reliability is actually associated with relatively low government losses for a range of parameters. In particular, expected losses are smaller than the case where the government provided even less reliable emergency financing.

The contribution of this analysis is to highlight that providing a limited form of emergency financing during a freeze may increase value. There are a number of features specific to our analysis and hence a number of caveats attach. First, we are focusing on a specific type of “bailout,” one that incentivizes debtors to roll over their debt during a freeze, and not other types of bailouts, such as nationalization or direct equity injection. Second, our model is agnostic as to the source of the emergency financing, which strictly speaking could be provided by entities other than the government. In this sense our point is that government-funded emergency financing may increase value if and when no one else is willing to provide it for reasons outside the model. We believe this is a reasonable starting point in light of the absence of credit in the crisis. Finally, our model does not explicitly model systemic risk, the
primary rationale advanced in favor of bailouts. We are essentially taking a reduced form approach: we take as given that there are large costs to liquidation and ask whether such policies aimed at avoiding such liquidation may enhance value. Such an analysis inherently cannot identify whether bailout policies affect managerial incentives to increase liquidation costs themselves, e.g., by becoming “too big to fail.” Nevertheless we believe our analysis to be a reasonable starting point and discuss these caveats in more detail in Section 5.

Although our model is similar in spirit to static bank run models such as Diamond and Dybvig (1983) and Goldstein and Pauzner (2005), and the bank-run/incentive models of Diamond and Rajan (2001) and Diamond and Rajan (2000), our dynamic model introduces two important insights. First, as noted, our dynamic model allows us to highlight the competing interests of maturing and non-maturing creditors and its interaction with incentives. We use this to show that it can be optimal to risk-shift during a freeze, where the volatility of a time-varying fundamental plays a role, much as it does in dynamic option-pricing theory, in improving creditor confidence by alleviating the concerns of future creditors (who have convex interests). Second, our analysis using a dynamic model allows us to compute the expected costs of a bailout within an equilibrium that allows for the occurrence (and recurrence) of freezes and risk-shifting in the future. This is in contrast to much of the existing literature where the costs and benefits of a bailout are evaluated by comparing the efficiency of various multiple equilibria, usually a “run” and “no-run” equilibrium. By using a dynamic model we are able to compute the expected costs of a bailout directly in an integrated framework accounting for its impact on present and future incentives and freezes.

The paper proceeds as follows. Section 2 introduces the model, and Section 3 describes our equilibrium and our results pertaining to optimal maturity and optimal risk-shifting. Section 4 examines the effect of emergency financing, and Section 5 discusses further implications of our analysis. Section 6 concludes.

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4One paper that studies the interaction between liquidity and incentives is Diamond (1991). In contrast to our setup, which has more to do with incentives and moral hazard, Diamond (1991) examines the cross-sectional determinants of maturity structure and explains why firms with higher credit ratings take on short-term debt and why firms with low credit ratings take on long-term debt or bank loans. Rochet and Vives (2004) study bailouts in a global games framework and have a section on incentives, but do not consider the dynamic feedback mechanism between risk-shifting and ‘runs’ or freezes.

5This result is reminiscent of Leland and Toft (1996) and Leland (1998). We discuss the relationship in Section 5.
2 The Model

The model is set in continuous time with \( t \in [0, \infty) \). There are two kinds of agents, on one side the manager who runs a firm and on the other side a continuum of identical creditors of said firm. Importantly, the creditors will split into two types, maturing creditors and non-maturing creditors at each time \( t \). We will look at a third party, the government, in Section 4.

2.1 The Firm

A non-bank financial firm holds an asset that generates a fixed continuous cash-flow \( r \) until the realization time \( \tau_\phi \) and a final payoff \( y_{\tau_\phi} \) at \( \tau_\phi \). The asset’s final payoff evolves according to

\[
\frac{dy_t}{y_t} = \mu_i dt + \sigma_i dZ_t
\]

where \( dZ \) is a standard Brownian motion, \( \mu_i \) is the growth rate of the project, and \( \sigma_i \) is the instantaneous volatility. The realization time \( \tau_\phi \) is exponentially distributed with intensity \( \phi \). We think of the asset as a long-term, illiquid investment held by the non-bank financial firm.

2.2 Creditors and Debt Contracts

The firm finances its holding of the asset using debt which has a staggered maturity structure. Debt in our model is most closely interpreted as asset-backed commercial paper. As noted in He and Xiong (2009a), many financial firms spread out the maturity of their debt expirations. Our model of creditor behavior builds upon theirs in that a continuum of risk-neutral creditors of unit measure holds debt of face value 1 each, which is needed to maintain the firm’s position in the asset. Debt receives the full cash-flow \( r \) that the project generates, so that the model is essentially cashless from the firm’s perspective. As creditors have a discount rate \( \rho < r \), debt is financially attractive to them.

Once a creditor lends money to the firm, the debt contract lasts a random time. A creditor’s contract comes due upon the arrival of an independent Poisson shock with intensity \( \delta \) where \( \delta \) measures the maturity mismatch between the debt structure and the actual payoff structure of the firm’s asset. When \( \delta = 0 \), the creditor is locked in until the firm’s final asset payoff is realized at \( \tau_\phi \). On the other hand, when \( \delta > 0 \), some debt comes due before the project realizes, with high values of \( \delta \) corresponding to very short-term debt structure.
These modeling assumptions lead to a staggered debt maturity structure - at any time \( t \), there is a \( \delta dt \) of *maturing* creditors (essentially senior creditors at that instance) and a measure \((1 - \delta dt)\) of *non-maturing* creditors (essentially junior creditors). Thus, there are 2 types of creditors, which we will have to treat separately. The firm may either be liquidated in distress, or the asset’s final payoff may realize while a creditor is holding her debt contract. We call either of these events - liquidation or payoff realization - a *horizon event*. Define the project’s horizon time \( \tau = \min \{ \tau_\phi, \tau_\theta \} \) as the minimum time for either the project realizing \( (\tau_\phi) \) or the project liquidating \( (\tau_\theta) \). The maturity structure of the firm’s debt, conditional on no horizon event, can be described by \( \delta e^{-\delta T} \), plotted in Figure 1. In the following, we will refer to debt maturity conditional on no horizon event simply as maturity if no confusion can arise.

When a creditor’s contract matures, he has the option to either receive the face value of 1 or roll over his debt at no additional cost, a choice he makes based on the current value of the asset. Let \( y^* \) denote the critical threshold below which creditors choose to stop rolling over.\(^6\) We will term the situation in which all creditors refuse to roll over their maturing debt a *rollover freeze*. During a rollover freeze, the company can draw on pre-established credit lines or government money to fill the gap on the balance sheet that is caused by the departing creditors. However, these credit lines or possible government bailouts are not perfect: when all maturing creditors in a \( dt \) period decide to pull out, there is a probability \( \theta \delta dt \) that the company cannot find financing. In this case, the company is liquidated in distress and its assets are sold at a fire sale discount. The parameter \( \theta \) measures the reliability of the firm’s credit lines or possible government bailouts - the higher the value of \( \theta \), the more likely the firm will be liquidated in distress.\(^7\) It is obvious that all else equal, if all maturing creditors stop rolling over, the probability of default is higher for lower maturities (i.e. higher \( \delta \)).

As discussed in He and Xiong (2009a), the important feature of this modeling device is that it captures intertemporal coordination problems between creditors - there is essentially seniority by maturation.\(^9\) Creditors may stop rolling over if they anticipate that future cred-

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\(^6\) We have a continuum of creditors, each of whom can choose a different threshold, but we look for a symmetric equilibrium among creditors. We thus use \( y^* \) to denote the threshold used by all creditors when no confusion can arise.

\(^7\) For now, the distinction between whether \( \theta \) represents a credit line or government money is not important; we return to this distinction in Section 4.

\(^8\) In a wider sense, part of the parameter \( \theta \) can also be interpreted as the strength of the company’s cash-holdings to overcome the possible criticism of the model as cash-less. Actually introducing cash reserves would result in a second state-variable that would make the model very hard to solve.

\(^9\) This modeling of rollover freezes is different from the intratemporal coordination problem which is com-
tions may stop rolling over and if firm fundamentals are unlikely to improve. The probabilistic
nature of the liquidation gives the realistic feature of being able to recover from a rollover
freeze.

The money from the credit lines is drawn under the standard debt contract described
above; that is, the debt acquired via a credit line is identical to the existing debt of the
company. The credit line providers immediately sell any such acquired debt on the secondary
market at a value $D$ to be determined later. When we interpret the distress financing to be
provided by the government, we assume the government uses a market based intervention:
it bails out those creditors that want to leave by paying them just enough to roll over their
debt.

Finally, we assume here that in times of distress, the firm cannot raise outside money due
to strong debt overhang or other corporate finance frictions, the terms of the debt contracts
are fixed, and there is no possibility of renegotiation.

[FIGURE 2 ABOUT HERE]

### 2.3 The Manager and Risk-Shifting

Up until now, the model follows He and Xiong (2009a). We depart by introducing the equity
side of the firm and a managerial incentive problem. A risk-neutral manager holds equity in
the firm, but does not hold any other form of wealth nor receive any non-stock salary. He
can either hold asset A, which is characterized by high expected return and low volatility,
or asset B, which has lower expected return but higher volatility. In other words $\mu_A > \mu_B$
but $\sigma_A < \sigma_B$ where $\{\mu_i, \sigma_i\}$ are the drift and volatility of each asset. The manager can
switch assets costlessly at any time based on current value of $y$. While these two investment
strategies - either hold A, or hold B - are quite stark, we view them as proxies for more
complex investment strategies, one of which reflects high expected value and low volatility,
and the other of which reflects low expected value and high volatility. In the absence of any
incentive considerations, it would be optimal for the manager to always hold asset A. Hence
we call asset A the “good” asset and asset B the “bad” or “risk-shifting” asset.

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mon in the static, two period banking literature. In this literature, there is usually a sequential service
constraint that works via the balance sheet: the first creditor gets to diminish the balance sheet by the full
nominal value of his claim, leaving less to be distributed to the remaining creditors. Thus, there is a direct
cost imposed by the creditor acting on the other creditors still waiting in line. In this model, a single creditor
running does not impose a direct cost on the other creditors, as the withdrawing of 1 unit of debt of measure
zero (recall the continuum assumption) does not instantaneously lead to liquidation or change the balance
sheet. Other creditors are affected by the shortening of the effective maturity (prepayment risk) and by
potential liquidation cost, both of which are stochastic.
Asset holdings cannot be contracted on by the debtors (even though which asset the manager is currently holding is observable via the quadratic variation of the observable process \( y \)). Similarly, the manager cannot commit ex ante to not risk-shift. We assume that the manager needs continued funding from debtors to finance the firm’s holdings. This gives rise to a risk-shifting problem: since the presence of debt makes equity a call option, the manager has an incentive to hold the bad asset when firm fundamentals are low (i.e., for low enough values of \( y \)). We consider this specific type of incentive problem as it is a particularly relevant concern for the recent financial crisis. For the remainder of the paper, we will refer to holding asset B as risk-shifting. We denote the region of values of \( y \) where the manager holds the bad asset with the set \( \mathcal{R} \).

### 2.4 Distressed Liquidation

The company’s asset-side of the balance sheet is simply made up of the project. If the firm is liquidated in distress, the project is sold on the outside market, where it fetches its risk-neutral value under asset A, i.e. its expected cash flow under A discounted by \( \rho \). This payoff stream is without the possibility of future liquidations and thus without incentive conflicts. The proceeds are then split equally among all non-maturing creditors. Thus, there is equal seniority of different debt ‘vintages’ in the sense that the last time the specific creditor rolled over is irrelevant.

However, there is a proportional cost of sale for distressed liquidations \( (1 - \alpha) \), with \( \alpha \in (0, 1) \) due to illiquidity of the underlying assets and/or bankruptcy costs. Thus, the project’s outside value is just the expected discounted cash-flows under asset A multiplied by \( \alpha \), i.e.

\[
L (y) \equiv \alpha \mathbb{E}^A \left[ \int_t^\tau \phi e^{-\rho(s-t)} r ds + e^{-\rho(\tau_\phi-t)} y_{\tau_\phi} \right] \\
= \frac{\alpha r}{\rho + \phi} + \frac{\alpha \phi}{\phi + \rho - \mu_A} y \\
= L + ly
\]

Even in the event of a distressed liquidation in which all creditors can be paid off there is still an aspect of prepayment risk as the cash-flow stream of \( r \) ceases.
2.5 Value Functions

We compute the expected value of debt and equity given a $y^*$ (the creditors' symmetric rollover threshold) and the risk-shifting region $\mathcal{R}$. For now, we characterize $\mathcal{R}$ as the union of two open intervals, $\mathcal{R} = (0, \bar{y}_1) \cup (\bar{y}_2, \bar{y}_3)$. We will discuss shortly why this makes sense. Let the function $D(\cdot)$ denote the value of the debt contract to a non-maturing risk-neutral creditor as a function of the current value of $y$. Then, we know that an agent who has maturing debt this instant will have a value function $\max\{1, D(y)\}$. Let the equity value function be denoted by $E(\cdot)$. As the manager is only compensated via equity, $E$ is sufficient to summarize the manager’s incentives. For both debt $D$ and equity $E$, $y^*$ and $\mathcal{R}$ are latent variables we will drop for notational convenience when no confusion can arise.

The value function for debt can be written as the following expectation

$$D(y) = \mathbb{E}_t \left[ \int_t^{\min\{\tau, \tau_0\} - t} re^{-\rho(s-t)} ds + 1_{\{\min\{\tau, \tau_0\} = \tau_0\}} e^{-\rho\tau} \max\{1, D\} + 1_{\{\min\{\tau, \tau_0\} = \tau_0\}} e^{-\rho\tau} \min\{L + ly_\tau - 1, 1\} \right]$$

where the first term is for the interest payment, the second term describes the rollover decision at maturity$^{11}$, the third term is the payoff from the project realizing and the last term is the payoff from liquidation.

Similarly, equity can be written as

$$E(y) = \max_{\{\text{asset}\}} \mathbb{E}_t \left[ 1_{\{\tau = \tau_0\}} e^{-\rho\tau} \max\{y_\tau - 1, 0\} + 1_{\{\tau = \tau_0\}} e^{-\rho\tau} \max\{L + ly_\tau - 1, 0\} \right]$$

where the first term is the payoff from the project realizing and the second term is the payoff from liquidation.

As the appendix shows, these value functions lead to HJBs that give rise to a system of ODEs for $D$ and $E$.

**Proposition 1** Given a rollover threshold $y^*$ and a risk-shifting region $\mathcal{R}$, possibly non-optimal, equity has the value function

$$E(y) = C_+(y) y^{\tilde{o}(y)} + C_-(y) y^{\tilde{o}(y)} + a(y) \cdot y + b(y)$$

$^{10}$Note the notational difference between the expectations operator $\mathbb{E}[\cdot]$ and the equity value function $E(\cdot)$.

$^{11}$As default/liquidation and individual rollover decision coinciding at a time $t$ is an event of bounded variation of order $dt$, we can ignore this event. Thus, the creditor’s only decision at time $\tau_0$ (the maturity time) is whether to rollover and get continuation value $D(y|y^*, \bar{y})$ or to collect the face value of 1.
where \( \eta(y)_+ > 1 > 0 > \eta(y)_- \) solve \( f(\eta, y) = 0 \),
\[
f(\eta, y) = \frac{\sigma_i^2}{2} \eta^2 + \left( \mu_i - \frac{\sigma_i^2}{2} \right) \eta - \left( \phi + \rho + 1_{\{y < y^*\}} \theta \delta \right)
\]
and
\[
a(y) = \frac{\mathbb{1}_{\{y < 1\}} \phi + \mathbb{1}_{\{y \nless y^*\}} \theta \delta l}{\rho + \phi + \mathbb{1}_{\{y \nless y^*\}} \theta \delta} - \left( \mathbb{1}_{\{y \nless R\}} \mu_B + \mathbb{1}_{\{y \nless R^c\}} \mu_A \right)
\]
\[
b(y) = -\frac{\mathbb{1}_{\{y < 1\}} \phi + \mathbb{1}_{\{y \nless y^*\}} \theta \delta (1 - L)}{\rho + \phi + \mathbb{1}_{\{y \nless y^*\}} \theta \delta}
\]
where different \( \eta, a, b, C_\pm \) apply in each of the intervals composed of the boundary points 0, 1, \( y_L \) and the rollover and risk-shifting thresholds \( y^*, \bar{y}_1, \bar{y}_2, \bar{y}_3 \). The different coefficients \( C_\pm \) solve a linear system stemming from value matching and smooth pasting at the transition points 1, \( \bar{y}_1, \bar{y}_2, \bar{y}_3, y^*, y_L \) and the boundary conditions \( C^0_- = 0 \) and \( C^\infty_+ = 0 \).

Similarly, debt has the value function
\[
D(y) = CC_+(y) y^{\kappa(y)_+} + CC_-(y) y^{\kappa(y)_-} + aa(y) \cdot y + bb(y)
\]
where \( \kappa(y)_+ > 1 > 0 > \kappa(y)_- \) solve \( \phi f(k, y) = 0 \),
\[
\phi f(k, y) = \frac{\sigma_i^2}{2} \kappa^2 + \left( \mu_i - \frac{\sigma_i^2}{2} \right) \kappa - \left( \phi + \rho + 1_{\{y < y^*\}} (\theta + 1) \delta \right)
\]
and
\[
aa(y) = \frac{\mathbb{1}_{\{y < 1\}} \phi - \mathbb{1}_{\{y \nless y^*\}} \theta \delta + \mathbb{1}_{\{y \nless y^*\}} \theta \delta l}{\rho + \phi + \mathbb{1}_{\{y \nless y^*\}} (\theta + 1) \delta} - \left( \mathbb{1}_{\{y \nless R\}} \mu_B + \mathbb{1}_{\{y \nless R^c\}} \mu_A \right)
\]
\[
bb(y) = -\frac{r + \mathbb{1}_{\{y < 1\}} \phi + \mathbb{1}_{\{y \nless y^*\}} \theta \delta (1 - L) + \mathbb{1}_{\{y \nless y^*\}} \delta (\theta L + 1)}{\rho + \phi + \mathbb{1}_{\{y \nless y^*\}} (\theta + 1) \delta}
\]
where different \( \kappa, aa, bb, CC_\pm \) apply in each of the intervals composed of the boundary points 0, 1, \( y_L \) and the rollover and risk-shifting thresholds \( y^*, \bar{y}_1, \bar{y}_2, \bar{y}_3 \). The different coefficients \( 12 \)Note that \( f(0, y) < 0 \) and that \( f(1, y) = \mu - \left( \phi + \rho + 1_{\{y < y^*\}} \theta \delta \right) < \mu - (\phi + \rho) < 0 \) by assumption on \( \mu_A \) and \( \mu_B \). Furthermore, note that \( \eta(y)_+ \eta(y)_- = \frac{\Delta}{2} < 0 \) (where \( a, c \) are quadratic equation coefficients \( ax^2 + bx + c \)), so that \( \eta(y)_+ > 1 > 0 > \eta(y)_- \).

\( 13 \)Where \( C^0_- \) is shorthand for \( C_-(y) \) with \( y \in (0, \min \{1, \bar{y}_1, \bar{y}_2, \bar{y}_3, y^*, y_L\}) \) and \( C^\infty_+ \) is shorthand for \( C_+(y) \) with \( y \in (\max \{1, \bar{y}_1, \bar{y}_2, \bar{y}_3, y^*, y_L\}, \infty) \).
CC± solve a linear system stemming from value matching and smooth pasting at the transition points 1, \( \bar{y}_1, \bar{y}_2, \bar{y}_3, y^*, y_L \) and the boundary conditions \( CC_0^0 = 0 \) and \( CC_\infty^\infty = 0 \).

The boundary conditions follow from some basic economic observations: \( C_\infty^\infty = 0 \) follows from the fact that equity cannot grow faster than the frictionless total value of the firm, which is linear in \( y \), i.e. \( \lim_{y \to \infty} E(y)/y < \infty \). Similarly, \( C_0^0 = 0 \) follows from the value function remaining bounded as \( y \to 0 \). \( CC_0^0 = 0 \) and \( CC_\infty^\infty = 0 \) follow from the observation that the payoff to debt is bounded for both \( y \to 0 \) and \( y \to \infty \), respectively.

Note that these are solutions with no inherent optimality or equilibrium properties yet - the above equations hold for arbitrary thresholds \( (y^*, \bar{R}) \).

### 2.6 Parameter Restrictions & Numerical Benchmarks

We need to impose a few additional parameter restrictions for the model to make sense and the value functions to be well defined.

First, we assume that \( L + l \leq 1 \), so that the project at \( y = 1 \) is worth less if liquidated than if it realized immediately. This is important to rule out the manager unilaterally liquidating the project to cash in on the promised interest flow to the creditors. Thus, unlike in Leland (1994) and Leland and Toft (1996), there is no endogenous default triggered directly by the manager.\(^{14}\) Let \( y_L \) be the point at which, if liquidated, the project just yields enough to pay off all creditors. From our previous assumptions, we have

\[
y_L \equiv \frac{1 - L}{l} \geq 1
\]

so that \( [y - 1]^+ \geq [L(y) - 1]^+ \).

Second, we assume \( \rho + \phi > r \) to ensure that there is an incentive to stop rolling over on the part of the debtholders (see 3.1 for details). This also results in \( L < 1 \). Combined with the previous restriction \( r > \rho \), we have

\[
\rho < r < \rho + \phi
\]

\(^{14}\)As the model nests a liquidation option, we need to consider the following question: Does the manager have an incentive to sell the project himself in the open market and thus impose “prepayment risk” on the debt-holders? His payout when the project realizes is \( (y - 1)^+ \), whereas his payout when the firm is sold is \( (L + ly - 1)^+ \). By selling the project today, the manager is able to raise cash that is related to interest payments \( r \) (summarized by coefficient \( L = \rho/(\rho + \phi) \)), which would otherwise go to the debt holders. Thus, for consistency, we need to check that \( E(y) \geq [L(y) - 1]^+ \). This holds for all cases treated in this paper.
Third, we require
\[ \mu_A < \rho + \phi \]
in order for the firm to have finite value. Note that neither the maturity parameter \( \delta \), nor the volatility parameters \( \sigma_A \) and \( \sigma_B \), nor the liquidation intensity \( \theta \) enter these parameter restrictions so far.

Fourth, we make the following technical assumption: \( \mu_A, \sigma_A, \mu_B, \sigma_B \) are such that the positive root of \( \frac{\sigma_i^2 \eta^2}{2} + \left( \mu_i - \frac{\sigma_i^2}{2} \right) \eta - (\phi + \rho + \theta \delta) \) is larger for asset A than for asset B. This constraint makes the risk-shifting problem non-trivial by ensuring that the manager will want to hold the bad asset for low \( y \)'s.

For our numerical solutions, we use the following annualized benchmark parameter values:

\[ r = .1, \rho = .05, \phi = .2, \theta = 1, \delta = 10, \mu_A = .05, \mu_B = 0, \sigma_A = .1, \sigma_B = .3, \alpha = .7 \]

that fulfill all our parameter restrictions above with \( y_L \approx 1.03 \). The expected maturity of a debt contract is \( \frac{1}{\delta} = .1 \) years or just above a month. The subjective discount rate is \( \rho = 5\% \), whereas debt has interest payments equal to \( r = 10\% \). The project has a drift of 5\% if technology A is used and 0\% if technology B is used. However, the annualized instantaneous volatility is three times larger for B, 30\%, than for A, 10\%. Finally, there are liquidation costs of 30\%.

Let us translate these intensities into probabilities and expected horizon times for better interpretation. When all maturing debtors run (and continue running), i.e., conditional on staying in the freeze, the expected time until the horizon event decreases from \( \frac{1}{\phi} = 5 \) years to \( \frac{1}{\phi + \theta \delta} = .098 \) years or 1.17 months - survival during a rollover freeze is difficult. Similarly, if a horizon event arrives during a freeze, there is a \( \frac{\delta \delta}{\phi + \theta \delta} = .98 \) probability that it leads to a distressed liquidation, and only a .02 probability of the project realizing. Thus, the liquidation event is by far the dominating event during a rollover freeze.

### 3 Rollover freezes, Risk-shifting and Maturity

We use the results from the previous section to look for a Markov-Perfect Nash equilibrium \( \{ y^*, R \} \) that arises when creditors symmetrically choose a rollover threshold to maximize their debt value (taking as given the manager’s risk-shifting strategy) and when the manager chooses a risk-shifting region to maximize his equity value (taking as given the strategy of the creditors). For an equilibrium, we need to have the individual creditor to be just indifferent
at \( y = y^* \) between rolling over his debt and receiving face value of 1. This equilibrium condition can be written as \( D(\{y^*\}|y^*, \mathcal{R}) = 1 \). Also, at each point \( \bar{y}_i \), the manager has to be indifferent between the incremental value of holding the good asset and the bad asset. This can be represented, if \( \bar{y}_i \) is distinct from \( y^*, 1, y_L \), as a so-called “super contact” optimality condition, which we derive in the appendix. We will denote the equilibrium thresholds as \( \bar{y}_{iAB} \) and \( y^*_{AB} \) when both assets A and B are at the managers disposal.

We can then summarize the equilibrium conditions.

**Equilibrium 1** A symmetric Markov-Perfect Nash equilibrium \((y^*, \mathcal{R})\) in cut-off strategies must simultaneously satisfy

1. (Creditor’s Indifference Condition) \( D(\{y^*\}|y^*, \mathcal{R}) = 1 \) with \( D(\{y\}|y^*, \mathcal{R}) \) strictly increasing in \( y \).

2. (Manager’s Optimality Condition) The cut-off point \( \bar{y}_i \) in \( E(\{y\}|y^*, \mathcal{R}) \) either satisfying the supercontact condition \( \lim_{y \uparrow \bar{y}_i} E_{yy}(\{y\}|y^*, \mathcal{R}) = \lim_{y \downarrow \bar{y}_i} E_{yy}(\{y\}|y^*, \mathcal{R}) \) or being a corner solution in the sense that \( \bar{y}_i \in \{1, y_L, y^*\} \), with no profitable deviation \( \bar{y}'_i \) available at any \( y \) in either case.

The rest of this section is devoted to understanding how equilibrium is affected by the structure of debt. We find that there is an optimal maturity structure that trades off risk-shifting versus rollover risk.

### 3.1 Incentives to Roll Over

Consider a maturing creditor’s problem. If he rolls over his debt, he is locked in for an expected time of \( \frac{1}{\delta} \), during which his interests are essentially junior to other maturing creditors. He will then be exposed to movements in \( y \) and possible distressed liquidations, the former being influenced by the investment decisions of the manager and the latter by the rollover decisions of the maturing creditors. To see how rolling over debt exposes a creditor to other creditor’s rollover decisions, note that non-maturing creditors would like a rollover freeze (and thus possibly a liquidation) at \( y \) if and only if \( D(y) < \min\{L + ly, 1\} \), whereas maturing creditors will stop rolling over for any \( y \) such that \( D(y) < 1 \). Thus, there is clearly a wedge between the incentives of maturing and non-maturing debtholders.

How much a maturing creditor is exposed to the manager’s investment decision and rollover decisions of other creditors is thus critically affected by how long they expect a firm to survive during a freeze. For any given \( \delta \), when \( \theta \) is high the firm has unreliable interim
financing and in a freeze is likely to be liquidated soon. Note that for any finite \( \theta \) and \( \delta \), the creditor will always attach some positive probability to being able to act again even if the firm is still experiencing a freeze.\(^{15}\)

To more fully understand the motivation to stop rolling over, we first review each creditor’s consideration in the absence of risk-shifting. This draws heavily from He and Xiong (2009a), but we review the key points of intuition here which are important for our subsequent analysis of risk-shifting. Suppose the manager can only invest in asset A. Denote the equilibrium rollover threshold in this situation by \( y_A^* \). Figure 3 plots \( y_A^* \) for different values of \( \delta \). In all of our graphs, a dashed red line identifies results of this “only rollover freezes” case. The equilibrium rollover threshold increases as maturity decreases (i.e. \( \delta \) increases). Even though a maturing creditor can expect to act sooner if debt maturities are short, he does not expect to act any sooner than other creditors, who also have short expected maturities.\(^{16}\) Since it is unlikely that the time-varying fundamental will improve significantly over short maturity windows, it is then likely that the fundamental will still be low when other creditors’ debt comes due. A maturing creditor then anticipates that other creditors will likely stop rolling over, and hence is more likely to stop rolling over himself. This dynamic coordination problem implies that each creditor will employ a higher rollover threshold, so that short maturities will ultimately lead to a higher equilibrium rollover threshold \( y_A^* \). Consequently, the incidence of distressed liquidations rises. The coordination problem depresses the expected ex-ante debt value, as shown in the graph. The figure also shows that ex-ante equity and thus total firm value, respectively, are decreasing in \( \delta \), since rollover freezes impose expected losses on the equity holders. In sum, shorter maturities worsen the rollover problem with no apparent benefits to anyone.

\[ \text{[FIGURE 3 ABOUT HERE]} \]

\(^{15}\)We show in the appendix that \( y^* = 0 \) (all maturing creditors always rolling over) is not an equilibrium nor is \( y^* \to \infty \) (all maturing creditors never rolling over). Intuitively, it is not an equilibrium for all maturing creditors to always roll over because for low values of the fundamental, the debt value will be low and some creditors would take their face value and stop rolling over. Similarly, never rolling over is not an equilibrium because when fundamentals are high, it is a profitable one-shot deviation to stay in instead of taking the face value, even when all other creditors are running.

\(^{16}\)More formally, the number of creditors that get to act between now and the next maturity time is independent of \( \delta \), as it is a product of the expected time \( \frac{1}{\delta} \) and the flow of maturing creditors per unit of time \( \delta \).
3.2 Incentives to Risk-Shift

The presence of debt in the firm makes equity a call option. Thus, for values of \( y < 1 \), the manager is “out-of-the-money” on this option. Risk-shifting can increase the equity option value by trading off fatter tails against a lower mean. But risk-shifting dilutes debt and involves using a dominated asset, thus generally leading to inefficient outcomes from a social planner’s perspective. In the case of no rollover risk (i.e. \( y^* = 0 \)), the manager risk-shifts for \( y \) below \( \bar{y} = 1.19 \), well before his option is “out-of-the-money,” in order to maximize his equity.

Rollover risk can discipline managerial risk-shifting: freezes introduce the likelihood of liquidation for low values of \( y \), thus lowering the value of holding the bad asset. In more detail, suppose debtors stop rolling over for \( y \) below \( y^* < 1 \). Before the freeze region is reached, that is on \((y^*, \infty)\), holding the bad asset will increase the chance of ending up in freeze region, and is consequently less desirable. The possibility of a freeze and distressed liquidation in the future disciplines the manager when the firm is still doing well. However, when the firm is experiencing a freeze, that is on \((0, y^*)\), the manager wants to hold the bad asset for sure. We show this formally in the appendix. A freeze dramatically lowers the option value to the manager since a distressed liquidation (and no payout to the manager since \( y_L > 1 \)) becomes very likely. This skews the trade-off between thicker tails versus lower drift even more in the direction of thicker tails, particularly so since his option is already out of the money.

Note that, given a rollover threshold \( y^* \), it is entirely possible for a manager to want to play a non-monotone strategy in the following sense. Suppose he risk-shifts (i.e., holds the bad asset) on the freeze region \((0, y^*)\), again for \( y^* < 1 \). The rollover threshold may not be high enough to completely eliminate the option value of holding the bad asset outside of a freeze, and thus he may hold the bad asset for some values of \( y \) above \( y^* \). However, for values of \( y \) very close to \( y^* \) (when a freeze is imminent), the volatility of the bad asset is likely to move the fundamental into the freeze region, which is a very bad state of the world for the manager since it is likely that his firm will be liquidated and he will receive nothing. Because of this he may actually hold the good asset for some “buffer” region above \( y^* \). Above this buffer region, his equity value is still low but fundamentals are far enough away from the freeze region, so holding the bad asset becomes optimal again. For yet higher values of \( y \), his option is safely in the money and he will hold the good asset. For these reasons we investigate risk-shifting strategies of the form \( \mathcal{R} = (0, \bar{y}_1) \cup (\bar{y}_2, \bar{y}_3) \), but we allow \( \bar{y}_1, \bar{y}_2, \bar{y}_3 \) to coincide, and we call risk-shifting outside of the freeze region preemptive risk-shifting. Note
that preemptive risk-shifting is strictly inefficient.

3.3 Equilibrium

Let \((y_{AB}^*, \mathcal{R}_{AB})\) denote the equilibrium rollover threshold and risk-shifting set. Because we need to consider the interaction between the freeze and risk-shifting regions, we have to simultaneously solve for \(y^*\) and \(\mathcal{R}\). As we have closed form solutions for debt and equity conditional on \((y^*, \mathcal{R})\), the optimality conditions reduce to solving a system of non-linear equations in the variables \(\bar{y}_{1AB}, \bar{y}_{2AB}, \bar{y}_{3AB}, y_{AB}^*\). We will focus on equilibria with \(\bar{y}_{1AB} = y^*\) as the manager will always risk-shift in the freeze. The other two risk-shifting thresholds \(\bar{y}_{2AB}\) and \(\bar{y}_{3AB}\) are then found via a super contact condition. Finally, we check via the \(E(\cdot)\) function that for any candidate equilibria \((y_{AB}^*, \mathcal{R}_{AB})\), there is no profitable deviation \(\bar{y}_i^*\) at any level of \(y\). In graphs, the results that refer to the full equilibrium are generally represented by solid blue lines or shading for risk-shifting regions, whereas dashed red lines refer to the “only rollover freezes” equilibrium described above.

Figure 3 plots the equilibrium thresholds as a function of \(\delta\). The rollover threshold \(y_{AB}^*\) is presented as the blue line whereas the risk-shifting set \(\mathcal{R}_{AB}\) is identified by the shaded area. The light red is for the portion that is preemptive risk-shifting and light blue for the risk-shifting that occurs in direct response to a freeze. In contrast to He and Xiong (2009a), we find that a shorter maturity structure can be beneficial.

There will be three regions of interest in terms of maturity. In the following discussions, note that high values of \(\delta\) are associated with short maturities and low values of \(\delta\) are associated with long maturities. First, for the range \(\delta \in (0, 0.5)\), we have a monotone strategy for the manager with preemptive risk-shifting at \(\bar{y}_{AB} > y_{AB}^*\). Second, for \(\delta \in (0.5, 19.25)\) we have two disjoint regions of risk-shifting by the manager, i.e. \((0, \bar{y}_{1AB}) \cap (\bar{y}_{2AB}, \bar{y}_{3AB}) = \emptyset\). The manager preemptively risk-shifts over a range of \(y\) above the rollover threshold, reverts back to the good asset close to the rollover threshold to create a buffer zone, and then risk-shifts again once in the freeze. Third, for the range \(\delta \in (19.25, \infty)\), the preemptive risk-shifting vanishes and we are left with risk-shifting only as a response to freezes, i.e. \(\bar{y}_{AB} = y_{AB}^*\). We will discuss these different ranges in turn.

3.4 Effects of changing maturity

Let us now consider the effects of shortening the maturity structure, i.e. increasing \(\delta\). This will have three effects. First of all, shortening the maturity structure makes runs more
potent, as the liquidation intensity during the freeze is $\theta \delta$. Thus, there is a direct effect on the manager’s incentives: for a given rollover threshold $y^*$, the manager will risk-shift less outside a rollover freeze. Secondly, the higher intensity also affects the creditors directly: for a given $y^*$ and $\bar{\mathcal{R}}$, debt is now worth less for any level of $y$. Third, there is a large indirect or equilibrium effect: As debt is now worth less due to the higher liquidation intensity during the freeze, the equilibrium $y^*$ and $\bar{\mathcal{R}}$ will shift, resulting in possibly non-monotone changes in overall firm value.

A surprising result from the equilibrium effect is that for all but the longest maturities, risk-shifting by the manager during the freeze is value-increasing for both the equity-holding manager and the debtholders. Effectively, non-maturing creditors are junior to maturing creditors and thus have more convex interests in that they want the firm to survive (at least long enough until they can act again) so that the firm’s assets are not liquidated in a distress situation. They thus prefer a higher volatility since it increases the chance of moving above the rollover threshold $y^*$ before $\tau$. In other words, dynamic debtholders also like to have things “shaken up” in the firm for low levels of $y$ as they cannot commit to rolling over once their debt matures.

Figure 3 reveals that, for all but the longest maturities (all but the lowest $\delta$), imposing a “good asset only” restriction on the manager actually lowers the ex ante value of debt and equity, as represented by the dashed lines in the figure. The implication here is that debt policy which restrict a manager’s investment choices to only the good asset are actually not optimal for short-maturity debt, in particular at the optimal maturity, since it eliminates the beneficial risk-shifting during a freeze.

For $\delta = (0, 0.5)$, the manager preempts the freeze by risk-shifting at $\bar{y}_{AB} > y^*_{AB}$. As $\delta$ increases, the risk-shifting threshold $\bar{y}_{AB}$ decreases. Rollover freezes mildly discipline the manager, but not enough to keep him from risk-shifting preemptively. In this region, the marginal impact of shortening the maturity is to discipline the manager’s risk-shifting. Consequently, we will describe this region as *disciplining region*.

For $\delta \in (0.5, 19.25)$, the manager’s optimal strategy is described by two disjoint risk-shifting regions. The manager preemptively risk-shifts on $(\bar{y}_{2AB}, \bar{y}_{3AB})$ which lies strictly above $y^*_{AB}$, and risk-shifts in response to a freeze on $(0, y^*_{AB})$. However, the manager reverts back to the ‘good’ asset A on $(y^*_{AB}, \bar{y}_{2AB})$. This region of holding asset A gives the manager a buffer zone to avoid the freeze region by sacrificing valuable volatility that could have been derived from asset B, as discussed in the previous section. The marginal impact of shortening the maturity in this region is to shrink the preemptive risk-shifting set while only mildly increasing the rollover threshold. We will also refer to this region as part of the
disciplining region.

For \( \delta \in (19.25, \infty) \), the manager’s preemptive risk-shifting vanishes and he only risk-shifts in response to a freeze, i.e. \( \bar{y}_{AB} = y_{AB}^* \). The marginal impact of shortening the maturity on this interval is purely to worsen the coordination problem and thus increase the rollover threshold \( y_{AB}^* \). As already discussed, during a freeze the manager has very strong incentives to risk-shift, so higher \( y_{AB}^* \) in this region will also increase the manager’s risk-shifting in lock-step. However, the positive effect of risk-shifting during the freeze is not enough to outweigh the negative effect of the worsening coordination problem. We will refer to this region as the rat-race region, as in this region shorter maturities only lead to more severe rollover freezes.

### 3.5 Optimal structure of debt

Suppose the initial value of the fundamental is \( y_0 = 1.17 \). In order to find the optimal structure of debt, we have to weigh the inefficiencies created by risk-shifting against the inefficiencies created by rollover freezes. It turns out that, for a wide range of parameterizations, the optimal maturity is the longest possible maturity that eliminates preemptive risk-shifting. That is, the optimal maturity is on the border between the disciplining region and the rat-race region.

The reasoning is as follows. The disciplining effect of rollover freezes on the manager depends on the effectiveness of freezes to impose costs on the manager. This is directly related to the maturity measure \( \delta \) and the reliability of interim financing \( \theta \). This disciplining power increases with \( \delta \) in the disciplining region, but for higher values of \( \delta \) - values in the rat-race region - incentives are already maximized and an increase in \( \delta \) only serves to make the dynamic coordination problem more severe and increase the likelihood of distressed liquidation. At the optimal maturity, rollover freezes have just enough power to keep the manager from risk-shifting preemptively. Any shorter maturity would impose excessive liquidation costs without leading to any improvement in disciplining the manager, while any longer maturity would lead to preemptive risk-shifting by the manager. Thus, the optimal maturity is found at the boundary between the disciplining region and the rat-race region. For our benchmark parameters, the optimal maturity is \( \delta = 19.25 \) or an average debt maturity of 19 days. Notably, it is not optimal to completely match the maturity of assets with the maturity of

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\(^{17}\)We have investigated a number of different initial starting values. For starting values of \( y \) outside of the freeze region, our results are qualitatively unchanged and hence we take \( y_0 = 1 \) for expositional purposes. The key here is that we are only considering cases where we start the firm outside of a freeze, which we believe is reasonable. Note that the equilibrium points \( (y_{AB}^*, R_{AB}) \) do not depend on any initial starting value: a Markov-Perfect Nash equilibrium is dynamically consistent for all values of \( y \).
liabilities.\textsuperscript{18}

\textbf{Result 1} There is an optimal maturity structure that trades of the incentives to stop rolling over by the creditors for shorter maturities against the incentives of the manager to risk-shift in the absence of freezes. This ex-ante optimal maturity is the longest possible maturity that eliminates preemptive risk-shifting. Completely matching the asset and liability maturities, in particular, is not optimal.

Furthermore, we previously highlighted that risk-shifting during a run may actually be beneficial due to its equilibrium effect on creditor confidence. On the other hand, preemptive risk-shifting (risk-shifting that occurs before a freeze) is an inefficient transfer from debt to equity. This is because preemptive risk-shifting does not alleviate coordination problems, but rather increases the likelihood of the fundamental entering the freeze region. Thus risk-shifting in a dynamic context has two faces: preemptive risk-shifting destroys value, while risk-shifting during a freeze actually enhances value.

\textbf{Result 2} For short maturities, risk-shifting during a freeze is beneficial as it leads to higher chance of escaping the freeze region before liquidation. Thus short-term debt should not contain covenants that restrict managerial investment decisions.

\section{Market-Based Emergency Financing}

The previous section left out one implicit 'claimant' to the firm’s balance sheet, the provider of interim financing during a rollover freeze. We now consider an interpretation where we think of the firm in the model as the broad financial sector and where the government provides this emergency financing via a market-based intervention, which we label a “bailout.” One may think of incentives in our model as incentives in the broad financial sector to hold risky assets, and a freeze in our model as a freeze in short-term credit markets. During such a freeze, the financial system, which relies on these short-term credit markets to finance long-term assets, experiences severe distress with probability $\theta \delta dt$. One can think of this event as the failure of Lehman Brothers or some other systemically important institution. Our goal is to address how bailouts might be designed so as to address the specific nature of rollover

\footnote{Previewing the next section, even if the bailouts/insurance of providing emergency financing is accounted for, we see that the occurrence of risk-shifting decreases costs at the long maturity end (below a $\delta$ of approximately 19.25), but increases costs at the short end of maturities. The scale of the expected costs, however, is quite small, so that the inclusion of expected costs in the definition of total value does not change any of our conclusions.}
freezes and credit freezes while preserving adequate incentives to avoid excessive risk-taking, which we interpret as holding the bad asset.

In our version of a bailout, the government uses a market-based intervention by paying maturing creditors who wish to leave $[1 - D(y)]$, which is just enough on the margin to incentivize them to roll over.\(^\text{19}\) However, these bailouts are not completely reliable in that the emergency financing may dry up, in which case the freeze creates severe distress in the financial system. The bailout reliability is thus parameterized by $\theta$, which the government commits to at time 0. We assume that the government cannot condition on what state the financial system is in when bailing out, but can only observe if a freeze is occurring and how much money is required to keep the maturing creditors invested. More broadly, we can interpret $\theta$ as randomizing between bailouts (e.g. Bear Stearns) and failures (e.g. Lehman Brothers).

In this context one can think of $\delta$ as measuring the maturity mis-match between assets and liabilities in the broad financial sector. A number of authors have attributed to the widespread usage of short-term debt to finance long-term illiquid assets as a proximate cause of the crisis, and our parameter $\delta$ captures this feature. We focus our discussion on the question of what type of emergency financing the government should provide given an observed maturity structure of the sector. That is, we fix $\delta$ and look for the optimal $\theta$ at time 0. A no-bailouts policy is then captured by $\theta \to \infty$, whereas a government that always bails out is equivalent to $\theta = 0$. A perhaps more intuitive interpretation is provided through the transform $P(\theta) = e^{-\theta}$. It gives the probability of survival for a continuous freeze of length \(\frac{1}{\delta}\). Bailing out with probability one then corresponds to $P(0) = 1$, whereas bailing out with probability zero corresponds to $\lim_{\theta \to \infty} P(\theta) = 0$.\(^\text{20}\)

The optimal bailout reliability $\theta$ maximizes the total \textit{ex ante} value $F$, which we define to be the total value of the system (debt plus equity) less any expected government losses. To compute expected government losses, we need to measure how often and how much the government is called upon to contribute for a given strategy $\theta$, which we can compute for any value of $y$. Denote by $G(\cdot)$ this expected cost to the government as a function of $y$, so that total \textit{ex ante} value is $F(y) = D(y) + E(y) - G(y)$. Since we fix the initial value at $y_0 = 1$, we will be interested in $G(1)$ and $F(1)$.

Whenever maturing creditors refuse to roll over, the government pays them just enough,\(^\text{19}\) An alternative interpretation of this bailout strategy is the following. The government purchases debt from distressed firms at face value 1. It then immediately turns around and sells this debt on the open markets for $D(y)$, making a loss of $[1 - D(y)]$ per unit of debt bailed out.\(^\text{20}\) If there is some level of private insurance in the form of credit lines, or buffer cash holdings, there will be an upper limit $\bar{\theta}$ that we ignore here for expositional purposes.
i.e. \([1 - D(y)]\), to roll over their debt. Thus, on \(y < y^*\), the government continuously provides emergency financing to a measure \(\delta dt\) of maturing creditors. The function \(G\) can then be written as

\[
G(y|y^*, \mathcal{R}) = \mathbb{E}_t \left[ \int_t^T e^{-\rho s} \delta 1_{\{y<y^*\}} \left[1 - D(y)\right] ds \right]
\]

We observe the following: First, in equilibrium, a freeze only occurs when debt is worth less than its face value, i.e. \(D(y) < 1\), which occurs for \(y < y^*\), and thus \(G(y) > 0\). Second, we must have \(\lim_{y \to \infty} G(y) = 0\), as the incidence of having to supply interim financing becomes negligible for large \(y\) as no freezes occur. The appendix shows the derivation of the associated HJB for the government, which we can solve to yield:

**Proposition 2** Given a rollover threshold \(y^*\) and switching thresholds \(\bar{y}_1, \bar{y}_2, \bar{y}_3\) the government’s bailout cost are

\[
G(y|y^*, \mathcal{R}) = CCC_+ y^{\eta(y)_+} + CCC_- y^{\eta(y)_-} + \delta 1_{\{y<y^*\}} \left[ \frac{-aa(y)}{\rho + \phi + \theta \delta - \mu} y + \frac{1 - bb(y)}{\rho + \phi + \theta \delta} \right]
\]

\[
+ \delta 1_{\{y<y^*\}} \frac{CC_+}{\sigma^2 \kappa(y)_+^2 + \left(\mu - \frac{\sigma^2}{2}\right) \kappa(y)_+ - (\rho + \phi + \theta \delta)} y^{\kappa(y)_+}
\]

\[
+ \delta 1_{\{y<y^*\}} \frac{CC_-}{\sigma^2 \kappa(y)_-^2 + \left(\mu - \frac{\sigma^2}{2}\right) \kappa(y)_- - (\rho + \phi + \theta \delta)} y^{\kappa(y)_-}
\]

where the appropriate \(aa, bb, \kappa_+, \kappa_-\) from the previous proposition apply in each of the intervals composed of the boundary points 0, 1, \(y_L\) and the rollover and risk-shifting thresholds \(y^*, \bar{y}_1, \bar{y}_2, \bar{y}_3\). The different coefficients \(CCC_{+/-}\) for each region solve a linear system stemming from value matching and smooth pasting at the boundary points. The appropriate boundary conditions are \(CCC^\infty_+ = 0\) and \(CCC^0_- = 0.21\)

### 4.1 The effects of bailouts

We have seen that rollover freezes discipline risk-taking. Thus removing the incentives to stop rolling over may not necessarily lead to higher total value when risk-shifting effects are included. In particular, the effect of bailouts on expected government losses is highly nonmonotonic due to interactions between risk-shifting and freezes, demonstrated by the panel in Figure 4 that plots \(G\) for a range of \(\theta\). For low bailout reliabilities, the incentive to stop

\[21\] Note that as \(\kappa_+ \neq \kappa_-\) on \(y < y^*\), we know that \(\frac{\sigma^2}{2} \kappa^2_+ + \left(\mu - \frac{\sigma^2}{2}\right) \kappa_+ - (\rho + \phi + \theta \delta) \neq 0\), so division by zero does not arise.
rolling over is very strong, as rolling over exposes a maturing creditor to the possibility that severe distress will be created by future maturing creditors.

High bailout reliabilities lower total value because incentives are not provided and managers preemptively risk-shift. This not only lowers value because the bad asset has low expected value, but also feeds back into higher expected government losses. Importantly, even though creditors employ a low rollover threshold for high bailout reliabilities, the government’s costs do not necessarily drop, as the freeze region might be more likely to be reached due to preemptive risk-shifting by the manager. Preemptive risk-shifting can be bad enough to outweigh the positive effects from bailouts from avoiding liquidations and instilling confidence.

Let us now look at the effect of increasing the bailout probability $P(\theta)$ (or decreasing $\theta$). Like in the discussion with maturity, there will be three effects. First, there is the direct disciplining effect on the manager: as the intensity $\theta \delta$ drops, the manager’s incentive to risk-shift increases. Second, there is the direct effect of avoiding inefficient liquidations during a freeze, which is a positive effect. Third, the second effect will make debt worth more for a given $y^*$. Maturing creditors will anticipate that future creditors are less likely to refuse to roll over and thus to create severe distress in the system. This alleviates the coordination problem and decreases the rollover threshold $y^*$. We call this indirect effect of the government’s strategy influencing the rollover threshold the confidence effect. In fact, in the absence of incentive considerations, it is a dominant strategy for the government to always bail out if government costs are equally weighted against debt and equity. But there is also a second side of the equilibrium adjustment: a lower $y^*$ leads to less disciplining of the manager and thus possibly more equilibrium risk-shifting, a negative effect.

Thus, again mirroring our discussion in Section 3.5, we have two regions, one where the marginal effect of increasing bailout reliability is value-enhancing and one where the marginal effect is value-destroying. Increasing the bailout reliability is value-enhancing as long as it alleviates the coordination problem without creating preemptive risk-shifting, described by the $P(\theta)$ to the left of the vertical line in Figure 4, and which we label the strong incentives region. In this region, incentives have been maximized, and increasing the probability of a bailout lowers the rollover threshold. In contrast, for the $P(\theta)$ to the right of the vertical line in Figure 4, which we label the weak incentives region, the increasing bailout reliability destroys total value as it creates pre-emptive risk-shifting while only decreasing the rollover threshold very slightly.

We summarize the findings in 2 results:
Result 3 **Very high bailout reliabilities increase incentives for managers to hold bad assets and can thus lead to higher chances of ending up in a rollover freeze, exacerbating government losses and lowering total value.**

Result 4 **Very low bailout reliabilities prevent managers from holding bad assets but worsen the dynamic coordination problem dramatically, which is suboptimal for total value and is associated with moderate government losses.**

4.2 Optimal Emergency Financing

Figure 4 demonstrates that there is an optimal level of emergency financing that trades off managerial incentives to risk-shift, the incentives of creditors to stop rolling over and the government costs of this strategy. If the government weighs creditors, equity holders and taxpayers equally, the optimal bailout reliability $\theta$ is found at the boundary between these two regions, and is characterized in our parameterization by most reliable bailout that avoids preemptive risk-shifting at $P(\theta) = .21$.

In particular, we note that at the optimal bailout reliability, expected government losses are actually very low, even lower than for less reliable bailouts, as less reliable bailouts increase freezes. Expected government losses at other intermediate bailout probabilities, such $P(\theta) = .5$, are actually higher, even though the rollover threshold is lower. This is because risk-shifting changes the probability distribution for $y$ as we just discussed.

We summarize the findings below:

Result 5 **The optimal bailout reliability is the most reliable bailout that avoids preemptive risk-shifting. By preventing preemptive risk-shifting, negative feedback effects from risk-shifting into freezes are avoided; by providing some bailouts, severe distress is less likely. This keeps total value high and expected government losses low.**

4.3 Private vs Public Supply of Insurance

Our interpretation of $\theta$ as purely arising from a government intervention is not required. Instead, we can have $\theta$ split into a firm’s financial strength in the form of cash reserves, market based insurance and government bailouts. It is important to note then that it is highly unlikely anyone but the government would be able to deliver very low $\theta$ at a risk-neutral value (or at all), since in our interpretation the insurance must be provided precisely when there is a large negative systemic shock. In a sense, the model leaves out the possible
tradeoff between private insurance that becomes more expensive as \( \theta \) decreases (and more insurance must be provided to large negative systemic shocks) and government intervention that usually entails other issues. Another way to model this is that private insurance can only push \( \theta \) down to a certain level. To push \( \theta \) even lower, the government has to intervene.

### 4.4 Robustness

Our result on the optimal bailout reliability is qualitatively similar to our result on debt structure, which is not surprising because bailouts essentially extend the expected survival time during a freeze, while short debt maturities decrease it. In our results on maturity, we varied \( \delta \) holding \( \theta \) fixed. This mixes two channels since \( \delta \) affects both the liquidation intensity \( \delta \theta \) and the flow of maturing creditors \( \delta \). Changing \( \theta \) in turn only affected the liquidation intensity \( \delta \theta \) while leaving the flow of maturing creditors untouched. As a robustness check, we thus isolate the channel that works though the flow of maturing creditors while leaving the liquidation intensity untouched. We will look at equilibria along the diagonal of \( \theta \delta = 10 \) on which our benchmark point lies. As graph 6 shows, there is an optimal maturity \( \delta \) (and thus an optimal bailout point \( \theta = \frac{10}{\delta} \)) at which the preemptive risk-shifting vanishes. We can clearly see that no bailouts, i.e. \( \theta \to \infty \), are suboptimal even when we lengthen the maturity to keep the liquidation intensity constant. Thus, neither a very short maturity-highly reliable bailout combination is optimal, nor is a long maturity-unreliable bailout combination optimal. Our results highlight that bailouts operate through a similar yet distinct channel from debt maturities in balancing the trade-off between ex ante incentives and ex post bankruptcy costs.

### 5 Discussion

Our model is stylized in that a number of frictions are assumed. Here, we discuss those assumptions, and discuss further policy implications of our analysis and its relationship to the bank run literature.

#### 5.1 Modeling Assumptions

First, our model relies heavily on a stylized friction in the interest rate of debt - it is fixed at \( r \). Of course, this is a simplification that we employ for tractability. The important content of this modeling device is to keep the interest rate from fully adjusting to reflect the firm’s situation. In reality there are situations, e.g., for Lehman Brothers, where no one was willing
to lend at any interest rate.

Second, we abstract from endogenous debt maturity in this model. It should be clear that holding all other creditors fixed, each individual creditor will want a lower maturity on his debt. This is the so called maturity rat-race that is discussed in more detail in Brunnermeier and Oehmke (2009) and also in He and Xiong (2009a). Our model addresses a very different question in that we ask whether certain maturities are optimal for firm value by investigating the outcome of coordination and risk-shifting problems. This allows us to look for the debt maturity that is optimal for the firm from an ex ante perspective, whereas papers with endogenous debt maturities (e.g., Brunnermeier and Oehmke (2009)) look for what debt maturities would arise naturally. We view these two questions as both important and complementary, but distinct.

Third, we also abstract away from a dynamic capital structure in this model, in particular the possibility that the firm raises new capital in times of rollover freezes or when the firm is close to a freeze. Although this would increase the chances of avoiding distressed liquidation, rollover freezes occur precisely because maturing creditors, who are at that moment to senior to non-maturing creditors, are withdrawing their money due to the anticipation that non-maturing creditors will do the same when their debt comes due. In these situations it seems a reasonable assumption that it is difficult to raise sufficient new capital (equity or debt) to significantly alleviate the coordination problem when there are large amounts of debt outstanding, as is the case for the largest financial firms. Furthermore, this “debt-overhang” problem would be in particular force because rollover freezes occur when $y \in (0, 1)$, so that new equity would be particularly hard to raise, and new debt would be difficult to issue, especially if there are covenants governing the issuance of debt in those situations. He and Xiong (2009b) model this conflict of interest in a rollover risk setting formally.

Lastly, we assume that the government commits to a fixed reliability parameter $\theta$ at the beginning of the world. Our model analyzes whether a particular bailout reliability is optimal ex ante, and has little to say about whether specific bailouts are optimal ex post. We view this as reasonable since this allows us to speak of ex ante value. Another concern is that the government does not play a strategy that allows it to condition on $y$, i.e., to choose a $\theta(y)$. We leave this for potential future analysis and only remark here that it will be difficult from a political economy perspective to tailor the bailout strategies to fundamentals as such action requires a lot of political flexibility.

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22We do allow the government to choose a perfectly reliable bailout strategy in which all firms get sufficient emergency financing. However, this turns out to be suboptimal in this model.
5.2 Types of risk-shifting

Our results highlight that risk-shifting during a freeze has a role in increasing value by alleviating the coordination problem among creditors. This arises because a freeze shrinks the total amount that both equity and debt can expect to receive in the event of a distressed liquidation. By holding the high volatility asset during a freeze, the expected total “size of the pie” is increased since it alleviates the coordination concern among creditors, who may have convex interests when they are locked in. Even though the drift of the bad asset is lower, when liquidation is likely, this cost is very small relative to the benefit.

The standard literature argues that managers “gamble for resurrection” when firm fundamentals are low and that this is value-destroying. Our results draw a distinction between preemptive risk-shifting, which is akin to the traditional concept of gambling, and risk-shifting during a freeze. In particular, our model implies that risk-shifting during a freeze can be optimal when liquidation costs are high. Comparing the “good asset only” equilibrium in Figure 3 (represented by the dashed lines) with the full equilibrium reveals that, at the optimal maturity (the vertical line), removing the manager’s ability to hold the bad asset reduces the value of debt plus equity from 1.38 to 1.24, or an economically large 10% loss, as represented in the lower left panel. Our point here is not to say that managers should seek wildly bad assets during a freeze, but rather to highlight that debt policy (and firm policies in general) should not be so stringent as to inadvertently prevent managers from holding assets or taking actions that would otherwise be deemed too risky or otherwise bad ideas during normal times, as desperate times may call for desperate measures.

The result that it may be optimal to choose the high risk strategy has the flavor of some of the results in Leland and Toft (1996) and Leland (1998). Similar to those papers, in our model there are implicit claimants beyond just debt and equity. First, one can think of the liquidation cost as a “claimant” in the sense that it receives $(1 - \alpha)$ of the expected discounted cash flows in the event of a distressed liquidation. Second, one can think of the credit line providers as a claimant to the firm as well. Risk-shifting increases the value of debt plus equity by transferring value from the liquidation cost to those two claimants. The effect on credit line providers is ambiguous, but is often small compared to the overall value gain on debt plus equity for moderate to low values of credit line reliability as measured by $P(\theta)$. The dashed line in the lower left panel of Figure 4, which represent value under a “good asset only” equilibrium, highlight that constraining the manager to a good asset is only optimal - from the perspective of debt plus equity plus credit line providers - for moderate and low values of $P(\theta)$. Viewed from another perspective, when liquidation costs are high, our result
is that it can be efficient to transfer value from the liquidation cost to the debt and equity by having the manager risk-shift during a freeze.

Most important is that, even when we define a claimant of the liquidation cost (e.g. law firms), this is not a zero-sum game. As the inefficient strategy B is chosen at times, although some value is shifted from the liquidation cost claimants towards the other agents in the model due to changes in incentives, the overall value of the firm shrinks. Thus, this model features real inefficiencies as opposed to contractual inefficiencies of splitting up fixed cash-flows.

Furthermore, our results offer potentially new insights as to why we may not always observe gambling when firm fundamentals are low. In our model, the manager plays a non-monotone strategy in equilibrium: he only gambles outside of a freeze when there is a sufficient buffer between the rollover threshold and the current firm fundamental. Inside of that buffer region, it is optimal for the manager to play conservatively by holding the high-mean, low-volatility asset in order to minimize the chances of incurring a rollover freeze.

5.3 Policy Implications of Bailouts

Our analysis bears important policy implications that we have discussed in developing the results. Rather than re-stating them all here, we discuss whether these implications bear any relationship with what is observed in the real world.

First, it is worth noting that our model simplifies on a number of dimensions. We take a reduced form approach to modeling systemic risk by simply assuming that there is a large cost to liquidation, captured via $\alpha$. The reduced form version of systemic risk in our model is flexible in that we can vary the liquidation cost $\alpha$ from the government’s perspective separately from the liquidation cost that debtors and equityholders perceive. For example, suppose the government views the failure of the firm as a huge value loss for the overall economy, so that they employ a specific $\tilde{\alpha} < \alpha$ discount to the distressed liquidation value of the firm. The manager and creditors employ the original $\alpha$ since they do not internalize the economy-wide impact of distressed liquidation. We find that this does not qualitatively affect our results. We can also extend the model by simply assigning a higher or lower weight to government losses relative to firm value in the total value function $F$. We find that the optimal bailout policy is robust to increasing the weight on government losses.

Second, we note that we are focusing on a specific type of bailout. In our bailout, the government is not nationalizing the firm. Instead, it is a targeted bailout aimed at alleviating the panic among creditors that could lead to distressed liquidation. In other words, it is
aimed at instilling confidence. We believe this has basis in programs such as the various li-
quidity facilities the Fed has implemented and also the asset guarantee programs the Treasury has implemented. Although these did not literally distribute money to creditors from the government, they were aimed at improving market confidence and avoiding panicked runs on systemically important institutions. Involving markets as much as possible and instilling confidence has been a recurring theme of the federal government’s bailout programs. On a broader level, we believe our modeling device sheds insight into the effect of bailing out some firms but not others. In particular, around the Lehman Brothers crisis, many commentators speculated that the Fed had to “make an example” out of Lehman by letting it fail. While our model is clearly not an ex post analysis of whether the Fed should have or should not have bailed out Lehman Brothers (and misses many important features of Lehman’s specific situation), allowing Lehman to fail is consistent with the government committing to an only imperfectly reliable bailout policy.

Our model thus highlights the role that the government can play in instilling confidence in markets by providing incentives for debtors to roll over their debt because it computes expected government losses accounting for dynamic feedback effects between incentives and freezes. In particular, the model works in the context of rollover freezes rather than bank runs. We believe that bank runs are highly important and similar in spirit to our model, and view our analysis as largely complementary to the large bank run literature.\textsuperscript{23} We add here that, while we acknowledge the importance of bank runs and runs due to lack of common knowledge (as in global games models), we believe that intertemporal coordination problems are also independently interesting, particularly in light of the possibly different prescriptions on how to resolve them. In a run that reflects a lack of common knowledge, one prescription for intervention would be to gather everyone in a conference room, share all private signals and agree on a course of action. Once common knowledge is established, the efficient action can be taken. This has basis in history as with the Fed’s intervention with LTCM. On the other hand, in a dynamic rollover freeze it is not a failure of common knowledge, but an intertemporal coordination problem, that generates the incentive to stop rolling over or ‘run’. The prescription in this case is very different: gathering everyone in a conference room will not work, as non-maturing creditors cannot commit to rolling over when their contracts mature. This generates a different role for bailouts that assure current maturing creditors that future maturing creditors are unlikely to create severe distress, but at the same time balance expected losses due to worsened incentives.

\textsuperscript{23}See, e.g., Diamond and Dybvig (1983), Goldstein and Pauzner (2005), and the bank-run/incentive models of Diamond and Rajan (2001) and Diamond and Rajan (2000), as well as Rochet and Vives (2004).
5.4 Implication for financial regulation

It should be clear from the discussion of the model that risk-shifting is not unequivocally bad. In abstract terms, the gain from risk-shifting essentially comes from the inefficient technology improving enough on the incomplete (or inefficient) debt contract to cancel out its inherent lower drift. The question we are interested in now is the following: what if the government has some monitoring technology at its disposal that would allow it to ban risk-shifting? We will consider two cases.

In the first case, the government can only monitor when it is involved in a bailout of the company. This might be for political economy reason as the political will to regulate firms that are not receiving tax payer money is not sufficient. But we know that the government will only bailout in distress. For a large range of parameter values, it is inefficient in equilibrium for the government to ban risk-shifting in a run. Recall from our discussion that risk-shifting during a run is actually value increasing. It is the preemptive risk-shifting before distress occurs that leads to inefficiencies.

In the second case, suppose the government is actually able to completely ban risk-shifting. We also see that comparing our firm value functions to the He and Xiong (2009a) specification (the dashed red lines in the graphs) that only a combination of banning risk-shifting and very long maturities will dominate the optimal maturity structure under risk-shifting. If regulation or monitoring is costly, this gain might be insufficient to warrant regulatory involvement.

6 Conclusion

So is short-term debt ultimately optimal? In this paper, we constructed a model of a non-bank financial firm which faces rollover externalities due to the use of staggered short-term debt (as in He and Xiong (2009a)) and introduced equity and incentive problems in the form of managerial risk-shifting. We also introduced expected government losses as a function of the reliability of emergency financing - which we termed a “bailout” - and analyzed its interaction with freezes and risk-shifting.

We found that the optimal maturity balances the trade-off between incentives and rollover risk, and that a purely matched maturity was not optimal due to incentive considerations. In particular, the structure of debt is such that maturities are long enough to alleviate freezes but avoid the case where the manager holds the bad asset outside of a freeze, which we termed preemptive risk-shifting. Additionally, we found that risk-shifting during a freeze can actually increase firm value. In particular, creditors anticipate that managers will hold the
bad asset during a freeze, so that risk-shifting actually alleviates creditors’ concerns about what other creditors will do in the near future. Thus, taking away the ability for managers to switch between the good and bad project actually decreases firm value for a wide range of maturities.

Because analyzing incentive problems, freezes, and maturities are intricately tied to the availability of emergency financing, we analyzed the effect of bailout reliability on total value in a context where we interpret the firm as representing the broad financial sector. We found that there is a non-zero optimal bailout reliability that improves total ex ante value. Bailout reliabilities that are too high lower total value because it leads managers to hold the bad asset very often, in particular even when outside of a freeze, although creditors’ coordination concerns are alleviated. This feeds back into more severe rollover freezes with high government losses. On the other hand, bailout reliabilities that are too low lead to too many freezes and government losses. The optimal bailout policy is just reliable enough to avoid managers holding the bad asset outside of a freeze, but is non-zero in that it provides money to creditors during a freeze in order to alleviate their coordination problem. In particular the optimal bailout reliability avoids preemptive risk-shifting. This optimum bailout reliability is actually associated with relatively low government losses for a range of parameters. This is because, by alleviating much of the creditor coordination problem without worsening incentives, creditors employ less aggressive rollover strategies and the government losses are small.
References


Rochet, J.-C., and X. Vives (2004): “Coordination failure and the lender of last resort: was Bagehot right after all?,” *Journal of the European Economic Association*, 2(6), 1116–1147.
A Appendix: Derivation of differential equations

When the project realizes, the payoff to the creditor is \( \min \{ y, 1 \} \). Rewrite this as \( \min \{ y, 1 \} = 1_{\{ y < 1 \}} (y - 1) + 1 \). Similarly, when the project is liquidated, the payoff to creditor is \( \min \{ L + ly, 1 \} = 1_{\{ y > y_L \}} (1 - L - ly) + L + ly \). With a slight abuse of notation, let \( y^* \) be a (possibly suboptimal) candidate symmetric rollover threshold. As a proportion \( \delta \) of debt contracts matures each \( dt \), liquidation has an intensity of \( \delta \theta \) in the rollover freeze \( y < y^* \).

In equilibrium, we know that at the rollover threshold \( y^* \), debt is worth \( 1 \), so that \( \max \{0, 1 - D \} = 1_{\{ y < y^* \}} (1 - D) \). We substitute \( 1_{\{ y < y^* \}} (1 - D) \) for \( \max \{0, 1 - D \} \) even off the equilibrium path, so that the creditors behave suboptimally for non-equilibrium \( y^* \). For a candidate symmetric equilibrium we then need to check that \( D (y|y^*, \bar{R}) \) is increasing in \( y \). The ODE can be rewritten as

\[
\rho D = \mu y D_y + \frac{\sigma^2}{2} y^2 D_{yy} + \phi (1_{\{ y < 1 \}} (y - 1) + 1 - D) + \theta \delta 1_{\{ y < y^* \}} \left( \max \{ L + ly, 1 \} - D \right) + \delta 1_{\{ y < y^* \}} (1 - D) + r
\]

\[
\Leftarrow \rho D = \mu y D_y + \frac{\sigma^2}{2} y^2 D_{yy} + \phi (1_{\{ y < 1 \}} (y - 1) + 1 - D) + \delta \theta \left[ 1_{\{ y_L < y < y^* \}} (1 - L - ly) + 1_{\{ y < y^* \}} (L + ly - D) \right] + \delta 1_{\{ y < y^* \}} (1 - D) + r
\]

where we substituted out the \( \max \{ \cdot \} \) function via the appropriate indicator functions. The first two terms on the right-hand side (RHS) are simply the Ito terms from the dynamics of the state variable \( y \). The third term is the payoff when the project matures, \( \tau_\phi \), whereas the fourth term is the payoff of the project being liquidated, \( \tau_\theta \). The fifth term is the payoff from the debt maturing, \( \tau_\delta \) (i.e. either rolling over or collecting the face value) and the last term is simply the interest rate.
For equity we have the following HJB

\[ \rho E = \mu_i y E_y + \frac{\sigma_i^2}{2} y^2 E_{yy} + \phi \left( \max \{ y - 1, 0 \} - E \right) + \theta \delta_1 \left( y < y^* \right) \left( \max \{ L + l_i y - 1, 0 \} - E \right) \]

\[ \begin{cases} \quad i = A & y \in \mathcal{R}_c^e \\ \quad i = B & y \in \mathcal{R} \\
\end{cases} \]

\[ \iff \rho E = \mu_i y E_y + \frac{\sigma_i^2}{2} y^2 E_{yy} + \phi \left( 1_{\{1 < y \}} (y - 1) - E \right) + \theta \delta \left( 1_{\{y, (i) \} < y < y^* \}} \left( L + l_i y - 1 \right) - 1_{\{y < y^* \}} E \right) \]

\[ \begin{cases} \quad i = A & y \in \mathcal{R}_c^e \\ \quad i = B & y \in \mathcal{R} \\
\end{cases} \]

where once again we substituted out the \( \max \{ \cdot \} \) function via the appropriate indicator functions. The first two terms on the RHS are once again simply the Ito terms for \( y \). The third term is the payoff of the project realizing, and the last term is the payoff from the project being liquidated.

\section*{B Appendix: Derivation of the super-contact condition}

Recall the value function definition:

\[ \rho E = \max_{A,B} \left\{ \mu y E_y + \frac{\sigma^2}{2} y^2 E_{yy} + \phi \left( \max \{ y - 1, 0 \} - E \right) + \theta \delta_1 \left( y < y^* \right) \left( \max \{ L + l_i y - 1, 0 \} - E \right) \right\} \]

For a given value function, then, the manager chooses A over B (instantaneously) when

\[ \mu_A y E_y + \frac{\sigma^2}{2} y^2 E_{yy} + \theta \delta_1 \left( y < y^* \right) \max \{ L + l_i y - 1, 0 \} > \mu_B y E_y + \frac{\sigma^2}{2} y^2 E_{yy} + \theta \delta_1 \left( y < y^* \right) \max \{ L + l_i y - 1, 0 \} \]

and B over A when the other way around. Note that A and B enter the max equation directly only through \( \mu_i \) and \( \sigma_i \), and indirectly through the value function \( E \). But suppose we are already at the optimum. Then we have no change in the value function for an instantaneous switching between A and B. Thus, only the direct impact matters, and we are left with the following boundary condition at \( \bar{y} \) from indifference between A and B:

\[ \mu_A \bar{y} E_y^{(A)} + \frac{\sigma^2}{2} \bar{y}^2 E_{yy}^{(A)} + \theta \delta_1 \left( \bar{y} < y^* \right) \max \{ L + l_i \bar{y} - 1, 0 \} = \mu_B \bar{y} E_y^{(B)} + \frac{\sigma^2}{2} \bar{y}^2 E_{yy}^{(B)} + \theta \delta_1 \left( \bar{y} < y^* \right) \max \{ L + l_i \bar{y} - 1, 0 \} \]

36
where the functions $E^{(i)}$ denote the value function with technology $i$ in use, i.e. $A$ applies to the right of $\bar{y}$, and $B$ to the left.

We can now derive the super-contact condition. Suppose that $y^* < \bar{y}$. Then, we have by the optimality of $\bar{y}$

$$
\mu_A \bar{y} E^{(A)}_y + \frac{\sigma_A^2}{2} \bar{y}^2 E^{(A)}_{yy} = \mu_B \bar{y} E^{(A)}_y + \frac{\sigma_B^2}{2} \bar{y}^2 E^{(A)}_{yy}
$$

$$
\mu_A \bar{y} E^{(B)}_y + \frac{\sigma_A^2}{2} \bar{y}^2 E^{(B)}_{yy} = \mu_B \bar{y} E^{(B)}_y + \frac{\sigma_B^2}{2} \bar{y}^2 E^{(B)}_{yy}
$$

as the conditions have to hold approaching from the right (i.e. for $E^{(A)}_y$) and from the left (i.e. for $E^{(B)}_y$) of $\bar{y}$ - the derivative does not change instantaneously when we switch strategies for a $dt$ period. Write $\Delta x = x_A - x_B$. Note that we have value matching and smooth pasting at $y = \bar{y}$. Subtracting the bottom equation from the top one, we can derive the super-contact condition:

$$
\frac{\sigma_A^2}{2} \bar{y}^2 \Delta E_{yy} = \frac{\sigma_B^2}{2} \bar{y}^2 \Delta E_{yy} \iff \frac{\Delta \sigma^2}{2} \bar{y}^2 \Delta E_{yy} = 0 \iff E^{(A)}_{yy}(\bar{y}) = E^{(B)}_{yy}(\bar{y})
$$

where the last line follows from $\bar{y} \neq 0$.

Suppose instead that $y^* > \bar{y}$. Then we have

$$
\mu_A \bar{y} E^{(A)}_y + \frac{\sigma_A^2}{2} \bar{y}^2 E^{(A)}_{yy} + \theta \delta 1_{\{y_L < \bar{y} < y^*\}} (L + l_A \bar{y} - 1) = \mu_B \bar{y} E^{(A)}_y + \frac{\sigma_B^2}{2} \bar{y}^2 E^{(A)}_{yy} + \theta \delta 1_{\{y_L < \bar{y} < y^*\}} (L + l_B \bar{y} - 1)
$$

$$
\mu_A \bar{y} E^{(B)}_y + \frac{\sigma_A^2}{2} \bar{y}^2 E^{(B)}_{yy} + \theta \delta 1_{\{y_L < \bar{y} < y^*\}} (L + l_A \bar{y} - 1) = \mu_B \bar{y} E^{(B)}_y + \frac{\sigma_B^2}{2} \bar{y}^2 E^{(B)}_{yy} + \theta \delta 1_{\{y_L < \bar{y} < y^*\}} (L + l_B \bar{y} - 1)
$$

Subtracting the bottom equation from the top one, we can again derive the super-contact condition. This of course only holds if the second derivative is continuous in $\bar{y}$. This can cease to hold at transitional points $1, y_L, y^*$ at which we can have asymptotes with a switch in sign.
C Appendix: Dominance strategies

We show that always rolling over and never rolling over cannot be equilibria. First consider a scenario where every maturing creditor rolls over, i.e. \( y^* = 0 \). We will consider a one shot deviation of a single maturing creditor, i.e. he can decide today if to rollover or not but will roll over in the future (if given the chance). As \( y \to 0 \), debt will be worth \( D = \frac{r}{\rho + \phi} < 1 \) by our parameter assumptions, whereas as \( y \to \infty \), debt will be worth \( D = \frac{r + \phi}{\rho + \phi} > 1 \). From the continuity of the value function we know that \( D \) crosses 1 at some point. It is below this point that the individual creditor will stop rolling over. But since the creditors are identical, they all have the same incentives and thus shift their rollover threshold up. We conclude that \( y^* = 0 \) cannot be an equilibrium.

Now suppose every maturing creditor never rolls over, i.e., \( y^* = \infty \). Again, we will consider a one shot deviation of a single maturing creditor, i.e. he can decide today if to rollover or not but will not roll over in the future. As \( y \to 0 \), the project will be worth \( D = \frac{r + \phi + \delta \theta L + \delta}{\rho + \delta \theta + \delta} < 1 \), so clearly for low levels of \( y \) it is never profitable to roll over the debt. But as \( y \to \infty \), debt is worth \( D = \frac{r + \phi + \delta \theta + \delta}{\rho + \delta \theta + \delta} > 1 \), so there will be a point at which the creditor will want to stay in the firm even if everyone else withdraws at the first chance they get. We conclude that \( y^* \to \infty \) cannot be an equilibrium either.

D Appendix: Derivation of \( C_{\pm}, CC_{\pm} \)

Wlog, consider a montone cut-off strategy by the manager, i.e. \( \vec{R} = (0, \bar{y}) \). Value matching and smooth pasting at transitional points, i.e. points that can be recrossed and are thus not absorbing, are properties of the value function that directly derive from its definition as a conditional expectation \( \mathbb{E}_t [\cdot] \). We will use the notation \( C_{k}^{\pm} \) with \( k \) denoting the left boundary of the interval in consideration with the exception of \( CC_{\pm}^{\infty} \) which denotes the last interval \((\max \{1, \bar{y}, y^*, y_L\}, \infty)\). This way we can write statements about \( C_{\pm}^{\infty} \) without the notational clutter implied by the interval or the possible ordering of \( \{1, y_L, \bar{y}, y^*\} \).

The different coefficients \( C_{\pm} \left(CC_{\pm}\right) \) for equity (debt) solve a linear system stemming from value matching and smooth pasting at the transitional points and the boundary conditions \( C_{-}^{0} = 0 \) (\( CC_{-}^{0} = 0 \)) and \( C_{+}^{\infty} = 0 \) (\( CC_{+}^{\infty} = 0 \)). Without loss of generality, consider the system of equations for a given interior solution \( \bar{y} \) and corresponding \( y^* \). Further, define \((y_1, y_2, y_3, y_4) = \text{Sort}\ (1, \bar{y}, y^*, y_L)\), where the operator \( \text{Sort} \) denotes the ordering operation on the vector such that \( y_1 < y_2 < y_3 < y_4 \). We then have the following system of equations for the value functions \( E_0, E_1, \) etc that each follow the functional form presented in the main
part of the paper, i.e. \( E_j (y, C_j^+, C_j^-) = C_j^+ y^n + C_j^- y^m + a_j y + b_j \) where \( a_j \) and \( b_j \) are the appropriate constants for \( y \in (y[j], y[j + 1]) \). Let \( E'_i (\cdot, \cdot, \cdot) \equiv E_{y,i} (\cdot, \cdot, \cdot) \). Then, we have

\[
\begin{bmatrix}
    E_0 (y1, C_0^+) \\
    E_1 (y2, C_a^+, C_a^-) \\
    E_2 (y3, C_b^+, C_b^-) \\
    E_3 (y4, C_c^+, C_c^-) \\
    E'_0 (y1, C_0^+) \\
    E'_1 (y2, C_a^+, C_a^-) \\
    E'_2 (y3, C_b^+, C_b^-) \\
    E'_3 (y4, C_c^+, C_c^-)
\end{bmatrix} =
\begin{bmatrix}
    E_1 (y1, C_a^+, C_a^-) \\
    E_2 (y2, C_b^+, C_b^-) \\
    E_3 (y3, C_c^+, C_c^-) \\
    E_4 (y4, C_4^-) \\
    E'_1 (y1, C_a^+, C_a^-) \\
    E'_2 (y2, C_b^+, C_b^-) \\
    E'_3 (y3, C_c^+, C_c^-) \\
    E'_4 (y4, C_4^-)
\end{bmatrix}
\]

which defines the coefficients \( C_j^\pm \) as functions of \((y1, y2, y3, y4)\). When \( \bar{y} = y^* \), we have only 3 connection points \((y1, y2, y3)\), but the same general form of value matching and smooth pasting applies.

E Appendix: Only rollover freezes

In He and Xiong (2009a) the project technology is fixed at A and there are no managerial incentive considerations. The only decision left in the model without project choice is for creditors to decide when to stop rolling over. The equilibrium will thus be solely determined by the debt function \( D \). We conjecture a cutoff Markov strategy with creditors refusing to roll over for \( y < y^* \). In our notation, if the manager always plays A the risk-shifting set is empty, \( \mathcal{R} = \emptyset \).

Corollary 1 (He,Xiong) For a given rollover threshold \( y^* \), the creditors value function will be

\[
D (y|y^*) \equiv D (y|y^*, \emptyset) = CC_+ y^{\kappa(y)_+} + CC_- y^{\kappa(y)_-} + aa (y) \cdot y + bb (y)
\]

where \( \kappa (y)_+ > 1 > \kappa (y)_- \) solve \( ff (k, y) = 0 \),

\[
ff (\kappa, y) = \frac{\sigma^2}{2} \kappa^2 + \left( \mu_A - \frac{\sigma^2}{2} \right) \kappa - (\phi + \rho + 1_{\{y < y^*\}} (\theta + 1) \delta )
\]

\( \kappa (y)_+ > 1 > \kappa (y)_- \) solve \( ff (k, y) = 0 \),

\[
ff (\kappa, y) = \frac{\sigma^2}{2} \kappa^2 + \left( \mu_A - \frac{\sigma^2}{2} \right) \kappa - (\phi + \rho + 1_{\{y < y^*\}} (\theta + 1) \delta )
\]
and

\[
aa(y) = \frac{1_{\{y<1\}}\phi - 1_{\{y_L<y<y^*\}}\theta\delta + 1_{\{y<y^*\}}\theta\delta}{\rho + \phi + 1_{\{y<y^*\}}(\theta + 1)\delta - \mu_A}
\]

\[
bb(y) = \frac{r + 1_{\{1<y\}}\phi + 1_{\{y_L<y<y^*\}}\theta\delta(1-L) + 1_{\{y<y^*\}}\delta(\theta+1)}{\rho + \phi + 1_{\{y<y^*\}}(\theta + 1)\delta}
\]

where different \(\kappa, aa, bb, CC_\pm\) apply in each of the intervals composed of the boundary points 0, 1, \(y_L\) and the rollover and risk-shifting thresholds \(y^*\). The different coefficients \(CC_\pm\) for each region solve a linear system stemming from value matching and smooth pasting at the transitional points 1, \(y_L\), \(y^*\) and the boundary conditions \(CC_0^- = 0\) and \(CC_\infty^+ = 0\) that is given in the appendix.

Let \(y_A^*\) denote the equilibrium rollover threshold in this scenario, so that the equilibrium condition is \(D(y_A^*|y_A^*) = 1\). He and Xiong (2009a)'s result actually goes further than the above corollary in that they can show existence and uniqueness of the symmetric equilibrium \(y_A^*\) analytically. First, note that for any finite, strictly positive \(y^*\), we have \(\lim_{y\to\infty} D(y|y^*) = \frac{r+\phi}{\rho+\phi} > 1\) and \(\lim_{y\to0} D(y|y^*) = \frac{r+\delta(\theta+1)}{\rho+\phi+\delta(\theta+1)} < 1\). Second, one can analytically show that \(W(y) = D(y|y)\) is increasing and only crosses 1 once (at \(y_A^*\)). Third, it is possible show that \(D(y|y_A^*)\) is strictly increasing and continuous in \(y\), so that individual optimality for refusing to roll over below the equilibrium threshold \(y_A^*\) is established. As our full fledged model is considerably more complex, this strategy of direct proof is not feasible. We will instead rely on numerical methods to establish equilibria \((y^*, \bar{R})\).

\section{Appendix: Optimality of risk-shifting during rollover freeze}

We show that risk-shifting during a freeze is optimal for the manager. We prove the statement for \(\bar{y} < y < y^*\) and \(0 < y < \bar{y} < y^*\). For all other \(y > y^*\), since there is a positive probability of reaching point \(y = y^*\), we can rely on the recursive formulation for optimality.

Consider an exogenous rollover threshold \(y^* < 1\). Also, for expositional clarity, and wlog, take \(\bar{y}_1 = \bar{y}_2 = \bar{y}_3 = \bar{y}\) and assume \(\bar{y} < y^*\) (i.e. there is no preemptive risk-shifting; but either way, this shouldn’t influence the incentives to risk-shift for other values of \(y\) as we have a recursive definition). Recall that \(y_L \geq 1\). We will look at the derivative of \(E(y|\bar{y}, y^*)\) w.r.t. \(\bar{y}\). After substituting in the appropriate constants \(C_-(y)\) and \(C_+(y)\) and some tedious algebra,
we get the following results. Here \( \eta_{1+} \) is shorthand for \( \eta_{+}(y) \) with \( y \in (0, \min \{ \bar{y}, y^*, y_L, 1 \}) \) and so forth.

For \( y \in (\bar{y}, y^*) \), we have

\[
\frac{\partial E(y|\bar{y}, y^*)}{\partial \bar{y}} = - \left( \frac{(\eta_{1+} - \eta_{2-}) (\eta_{1+} - \eta_{2+}) (\eta_{2-} - \eta_{2+}) \phi (\phi + \rho - \eta_{3-} - \eta_{3+})}{\rho \eta_{n2} + \rho_2 - 1 (y^*)^{\eta_{n2} - (y^*)^{\eta_{n2} + (\eta_{3-} - \eta_{2+}) y^{\eta_{n2} - (y^*)^{\eta_{n2}}})}} \right)
\]

\[
\cdot \left( \eta_{2+} \cdot \eta_{3-} \cdot \bar{y}^{\eta_{n2} - (y^*)^{\eta_{n2}} - \eta_{2-} \cdot (\eta_{2+} \cdot \bar{y}^{\eta_{n2} - (y^*)^{\eta_{n2}} - \eta_{2+} \cdot \bar{y}^{\eta_{n2} - (y^*)^{\eta_{n2}} + \eta_{3-} \cdot \bar{y}^{\eta_{n2} - (y^*)^{\eta_{n2}}}) \right)^2
\]

Note that \((\eta_{1+} - \eta_{2-}) > 0, \eta_{3-} < 0, (\eta_{3-} - \eta_{2+}) < 0 \) and \((\eta_{2-} - \eta_{3-}) < 0\) directly from their definitions. Also by our assumptions in section 2.6, we have \((\eta_{1+} - \eta_{2+}) < 0\) (4th assumption) and \((\phi + \rho - \mu_A) > 0\) (3rd assumption). We conclude that this expression is positive for all \( y \in (\bar{y}, y^*) \).

For \( y \in (0, \bar{y}) \), we have

\[
\frac{\partial E(y|\bar{y}, y^*)}{\partial \bar{y}} = - \left( \frac{(\eta_{1+} - \eta_{2-}) (\eta_{1+} - \eta_{2+}) (\eta_{2-} - \eta_{2+}) \phi (\phi + \rho - \eta_{3-} - \eta_{3+})}{\rho \eta_{n2} + \rho_2 - 1 (y^*)^{\eta_{n2} - (y^*)^{\eta_{n2} + (\eta_{3-} - \eta_{2+}) y^{\eta_{n2} - (y^*)^{\eta_{n2}}})}} \right)
\]

\[
\cdot \left( \eta_{2+} \cdot \eta_{3-} \cdot \bar{y}^{\eta_{n2} - (y^*)^{\eta_{n2}} - \eta_{2-} \cdot (\eta_{2+} \cdot \bar{y}^{\eta_{n2} - (y^*)^{\eta_{n2}} - \eta_{2+} \cdot \bar{y}^{\eta_{n2} - (y^*)^{\eta_{n2}} + \eta_{3-} \cdot \bar{y}^{\eta_{n2} - (y^*)^{\eta_{n2}}}) \right)^2
\]

By the same inequalities mentioned above, the expression is positive for all \( y \in (\bar{y}, y^*) \).

What remains to be shown is that these assumptions are truly wlog, i.e. consider a strategy by the manager that has risk-shifting on an interval \((\bar{y}_2, \bar{y}_3)\) outside of \((0, y^*)\). Clearly risk-shifting in the rollover region gives lower liquidation values conditional on default (as the drift \( \mu_B \) is lower than \( \mu_A \)). Thus, the benefit of risk-shifting must come from increasing the probability of escaping the rollover-region and thus increasing the probability on the non-freeze value function. But if the value function outside of the freeze region can be improved by risk-shifting on an interval \((\bar{y}_2, \bar{y}_3)\), then clearly the trade off between liquidation value today and continuation value tomorrow becomes even more skewed towards the continuation value tomorrow, and thus the incentives of the manager are even stronger in favor of risk-shifting.
on the freeze region. QED.

G Appendix: Risk-shifting can be beneficial to creditors

We now show that it can actually be beneficial for the non-maturing debtholders to have a non-zero \( \bar{y} \) in the presence of rollover risk. To this end, fix \( y^* \) and look at the value function at \( y = y^* \) in the presence of risk-shifting and without risk-shifting. If we can show that debt is worth more in the case of risk-shifting than without, we are done.

\[
D(y^*|\bar{y} = y^*, y^*) - D(y^*|\bar{y} = 0, y^*) = 
\]

\[
y^*
\]

\[
+ \left( \frac{1}{\kappa_{1+}} \right) \frac{1}{\kappa_{1+}} \left( \phi + \rho - \kappa_{2-} \mu_A \right) \left( (y^* y_{k2+} - y^* y_{k2-}) \right)
\]

Plugging in our benchmark parameters, we see that this holds for a wide range of \( y^* \) as shown in figure 7. As we prove optimality of risk-shifting at the point \( y = y^* \), and the dynamics are the same for all \( y > y^* \), we also know that the nonmaturing debtholders will prefer risk-shifting at the point \( y_0 = 1 \). No such straightforward statements can be made for points in the freeze region. As the \( y^* \) was fixed, we can make the following statement. If the \( y_{1+}^* \) was optimal in the case without risk-shifting, such that \( D(y^*|\bar{y} = 0, y^*) = 1 \), then if debt is increasing we know that the optimal rollover threshold in case of risk-shifting \( y_{1+}^* \) will be below \( y_{1+}^* \). This will lead to an even higher utility gain to the creditors than shown in the graph.

[FIGURE 7 ABOUT HERE]
H Appendix: Bailouts

The associated HJB for the government’s loss function is

$$\rho G = \mu y G_y + \frac{\sigma^2}{2} y^2 G_{yy} - \phi G - 1_{\{y < y^*\}} \theta \delta G + \delta 1_{\{y < y^*\}} [1 - D(y)]$$

where the first two terms on the RHS are once again the Ito terms of $y$, the third term reflects the intensity of realization, the fourth term the intensity of default without a credit-line/bailout and the last term the liquidity injection stemming from the continuous bailouts on $(0, y^*)$. The equation is straightforward to solve - it is a linear ODE with 2 independent solutions from the quadratic equation, plus a particular part tied to the expression $D(y)$ that is available in closed form. Note that $G$ has the same fundamental equation as $E$, thus we use the same parameters $\eta_{\pm}$.

I Appendix: Supplementary graph

Figure 8 supplies the equity, debt and government value functions, as well as total firm value function for the benchmark of $\delta = 10$. The vertical lines reflect $y^*_{AB} = \bar{y}_{1AB}$ and $\bar{y}_{2AB}, \bar{y}_{3AB}$, the risk-shifting thresholds. The dashed lines are the lower bounds $[L(y) - 1]^+$ for debt value and $[y - 1]^+$ for equity values.

J Appendix: Single creditor with demand deposits

To gain some intuition on the run externality, let us consider the case in which a single creditor holds all the debt as demand deposits (i.e. he can at any time pull out all his money) and there is no risk-shifting technology available. This is to establish a comparison value to the $\delta = \infty$ case.

It is straightforward to show that the debt value function is given by

$$D(y) = CC_+(y) y^{\eta_+} + CC_-(y) y^{\eta_-} + aa(y) y + bb(y)$$

where $\kappa_+ > 1 > 0 > \kappa_-$ solve $ff(k) = 0$,

$$ff(k) = \frac{\sigma_A^2}{2} k^2 + \left(\mu_A - \frac{\sigma_A^2}{2}\right) k - (\phi + \rho)$$
and

\[ aa(y) = \frac{1_{\{y<1\}}}{\rho + \phi - \mu_A} \]
\[ bb(y) = \frac{r + 1_{\{1<y\}}}{\rho + \phi} \]

and where \( y^* \) if an interior solution exists is defined by a value-matching and smooth pasting condition w.r.t. \( L(y) = \min \{L + ly, 1\} \). Consider the corner solution \( y^* = 0 \). In this case, \( CC_-(0) = 0 \) and we must have \( CC_+(0) < 0 \) as otherwise we would just be pricing an all equity firm. At zero, the slope of \( L(y) \) is simply \( l = \frac{\phi}{\rho + \phi - \mu_A} \) whereas the slope of the debt function is \( \frac{\phi}{\rho + \phi - \mu_A} \) and thus larger. Secondly, in the vicinity of \( y = 0 \), the debt function is concave by \( CC_+(0) < 0 \). From value matching and smooth pasting at \( y = 1 \), we can solve for the coefficients

\[ CC_+(0) = \frac{\phi (\kappa - \mu_A - \rho - \phi)}{(\kappa_+ - \kappa_-) (\phi + \rho) (\phi + \rho - \mu_A)} \]
\[ CC_-(1) = \frac{\phi (\kappa + \mu_A - \rho - \phi)}{(\kappa_+ - \kappa_-) (\phi + \rho) (\phi + \rho - \mu_A)} \]

From the value matching condition at \( y = 1 \), we know that \( CC_+(0) = CC_-(1) + \frac{\phi}{\rho + \phi - \mu_A} - \frac{\phi}{\rho + \phi} \) so \( CC_+(0) < CC_-(1) \). Debt does not change functional form for \( y \geq 1 \), and we know that it asymptotes \( \frac{r + \phi}{\rho + \phi} \). We also know that debt is concave on \( y \in (0, 1) \). Consider now the case where

The debt value at \( y = 1 \) is

\[ D(1) = CC_-(1) + \frac{r + \phi}{\rho + \phi} \]
\[ = \frac{\kappa_+ \mu_A \phi - \phi^2 - \phi \rho + (\kappa_+ - \kappa_-) (r + \phi) (\phi + \rho - \mu_A)}{(\kappa_+ - \kappa_-) (\phi + \rho) (\phi + \rho - \mu_A)} \]

Plugging in our assumptions on coefficients, we see that \( D(1) = 1.187 > 1 \geq L(1) \). As debt is strictly increasing in \( y \) and strictly concave on \( y \in (0, 1) \), we conclude that \( D(y) > L(y) \forall y \in (0, 1) \), and because debt is strictly increasing, we also conclude that \( D(y) > L(y) \forall y \in (1, \infty) \). Thus, any deviation of liquidating the firm at any point \( y^* \geq 0 \) will be non-optimal. This shows that if the maturing debtholders were appropriately incorporating the liquidation externality they impose on non-maturing debtholders into their own decision making, the firm would never be liquidated.
Figure 1: Expected maturity distributions conditional on no horizon event.

Figure 2: Example paths of $y$ with possible outcomes at horizon times.
Figure 3: Top left: Optimal risk-shifting region $\tilde{R}_{AB}$ in grey, rollover threshold $y^*_A$ in blue and rollover threshold $y^*_B$ in red. Top right: Debt values $D$ at $y_0 = 1$ for only A in red and A and B in blue. Bottom right: Equity values $E$ at $y_0 = 1$ for only A in red and A and B in blue. Bottom left: Equity plus debt $E + D$ for only A in red and A and B in blue. The vertical line marks the optimum maturity $\delta$, with the region to the left the disciplining region and the region to the right the rat-race region.
Figure 4: Top left: Optimal risk-shifting region $\bar{R}_{AB}$ in grey, rollover threshold $y^*_{AB}$ in blue and rollover threshold $y^*_A$ in red. Top right: Debt values $D$ at $y_0 = 1$ for only A in red and A and B in blue. Bottom right: Expected bailout costs $G$ to government at $y_0 = 1$ for only A in red and A and B in blue. Bottom left: Firm value $F$ for only A in red and A and B in blue. The vertical line marks the optimum bailout reliability $\theta$, with the region to the left the rat-race region and the region to the right the disciplining region.
Figure 5: Firm value $F(1) = D(1) + E(1) - G(1)$ (blue, lower line) and $D(1) + E(1)$ (red, upper line) for $\delta$ and $\theta$. The shaded region is the difference, $G(1)$.

Figure 6: Firm value $F(1) = D(1) + E(1) - G(1)$ along the diagonal $\theta\delta = 10$. 

48
Figure 7: Difference in debt value at $y = y^*$ between $\bar{y} = y^*$ ($D^B$) and $\bar{y} = 0$ ($D^A$).

Figure 8: Equity (blue), debt (red), bailout costs (green) and overall firm value $F$ (thick blue) as a function of $y$ for the benchmark case $\delta = 10$. The vertical lines are $y^*_{AB} = \bar{y}_{1AB}$, $\bar{y}_{2AB}$ and $\bar{y}_{3AB}$ from the right. Note that $D(y^*|y^*, \bar{y}) = 1$. 

49