A Monopolistic and Oligopolistic Stochastic Flow Revenue Management Model

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This paper studies a one-shot inventory replenishment problem with dynamic pricing. The customer arrival rate is assumed to follow a geometric Brownian motion. Homogeneous customers have an isoelastic demand function and do not behave strategically. We find a closed-form optimal pricing policy, which utilizes current demand information. Under this pricing policy the inventory trajectory is deterministic, and a retailer sells all inventory. We show that dynamic pricing coordinated with the inventory decision achieves significantly higher profits than does static pricing. Furthermore, under oligopolistic competition we establish a weak perfect Bayesian equilibrium for the price and inventory replenishment game. We find the pricing equilibrium to be cooperative even in a noncooperative environment, but that inventory competition results in overstock and damages profits. Finally, we examine the trade-off between dynamic pricing and price precommitment and find that flexible pricing is still beneficial, provided competition is not too intense.

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1. Introduction and Literature Review

Advances in information technology have made it possible to track sales and inventory, as well as adjust production and pricing levels, more rapidly than ever. Many industries, such as airlines, hotels, and various retailers, use dynamic pricing to match demand with capacity or inventory, maximize revenue, or achieve other strategic goals (see, e.g., Talluri and Van Ryzin 2004 for more applications). In response, recent years have witnessed the growth of a dynamic pricing literature (see Bitran and Caldentey 2003 and Elmaghraby and Keskinocak 2003 for reviews). Because of the complex nature of dynamic pricing (mainly due to the cumbersome recursive structure of dynamic programming representations), much of this literature has relied heavily on numerical optimization. Closed-form optimal pricing strategies are still rare (Gallego and van Ryzin 1994, §2.3 provides an excellent exception). Although numerical techniques can be adapted to many different settings, they are not well suited to deriving general managerial insights. Moreover, it is difficult to combine numerical approaches to dynamic pricing with other operations decisions, such as production or procurement, or with a competitive framework. In this paper, we propose an analytically tractable pricing model that incorporates an inventory decision as well as competition.

We study dynamic pricing in the context of a one-shot inventory replenishment problem, in which the customer arrival rate is assumed to follow a geometric Brownian motion. Most of the dynamic pricing literature has modeled customer arrivals as a Poisson process (see, e.g., Gallego and van Ryzin 1994), which implies that demands are independent across intervals. Under this assumption, current demand information is not helpful for inferring future demand, and therefore there is no need to combine dynamic pricing and demand forecasting. However, in reality, demands often show a correlation structure, and hence demand information is valuable in dynamic pricing. Clearly, the simplification of the independent increment assumption of a Poisson process presents a serious obstacle to evaluating the merits of dynamic pricing as a tool for hedging demand uncertainty and allocating limited resources. Unlike a Poisson process, a geometric Brownian motion is Markovian and partially captures demand correlation, so using it in place of a Poisson process causes pricing policies to depend on current demand information and permits integration of demand forecasting with pricing.

Geometric Brownian motion is widely used in the finance literature and recently has been adopted in the operations management literature. Caldentey and Wein (2006) modeled spot prices as a bounded and piecewise-linear version of a geometric Brownian motion. Chod and Rudi
(2003) assumed that a forecast evolution process follows a geometric Brownian motion. There is also precedent for using general diffusion processes to model demand flows (see, e.g., Harrison et al. 1983 and Harrison and Taksar 1983). Raman and Chatterjee (1995) studied a dynamic pricing problem of a monopolist, in which sales are modeled by a diffusion process. Their focus was on the diffusion/saturation effects of demand, and experience curve effects of cost. Sapra and Jackson (2004) modeled buyer demand as a diffusion process and derived closed-form optimal capacity, production, and price trajectories. They assumed that no agents have market power and that customers have rational expectations. Our model assumes that retailers have market power to determine prices, but we avoid modeling strategic customer behavior and multiple inventory replenishments.

In addition to assuming nonstrategic customer behavior, we assume homogeneous customers who have an isoelastic demand function. Because of its useful log-linearity property, isoelastic demand functions are widely used in empirical economics and marketing studies (see Monahan et al. 2004 for other merits). Coupling isoelastic demand with a geometric Brownian motion gives us a closed-form solution of the optimal dynamic pricing policy. It is interesting to note that this optimal pricing policy forms a martingale. As such, our result can be considered a “random” version of the result by Gallego and van Ryzin (1994, §3.2), in which the optimal pricing policy is constant for a deterministic customer arrival process. Moreover, because isoelastic demand has decreasing revenue in price, by applying the optimal pricing policy the inventory path is deterministic (i.e., all uncertainty is absorbed by pricing), and a retailer sells all inventory.

Monahan et al. (2004) considered a multistage model with independent customer arrivals. At every stage, a price is set before the number of customer arrivals at that stage is revealed. They formulated a dynamic program, which has to be solved numerically. By comparing the value of pricing flexibility (recourse) with the one-shot pricing model of Petruzzi and Dada (1999), they demonstrated the advantage of dynamic pricing. In this paper, by allowing an infinite number of price changes and switching to Markovian arrivals, we are able to obtain a closed-form pricing policy. In Xu and Hopp (2005a), we extended the framework of Monahan et al. (2004) to allow serial correlation in demand by switching to responsive pricing and derived a semiclosed form optimal pricing policy. We compared it with two heuristic pricing policies and found that separating demand forecasting from dynamic pricing results in systematic overpricing and a downward price trend. This paper goes beyond Xu and Hopp (2005a) by considering continuous-time pricing and oligopolistic competition.

In the competitive case, we assume that multiple retailers sell a single product. Customers select a retailer according to a stochastic process that is independent of the customer arrival rate process and allows purchases only from retailers with the lowest price and positive inventory. We model the dynamic pricing competition process as a differential game (see, e.g., Başar and Olsder 1982). Differential games are widely used in economics and marketing areas for modeling capital accumulation, R&D, and advertising (see, e.g., Dockner et al. 2000 and Jørgensen and Zaccour 2004). As far as we know, however, they have rarely been applied in operations management. By assuming that retailers know the aggregate inventory level, we establish a weak perfect Bayesian equilibrium that coincides with a public perfect equilibrium (see, e.g., Fudenberg et al. 1994); that is, retailers form their pricing strategies based on publicly available signals and cooperate to achieve the system optimum. However, we find that this leads to significant overstocking in the inventory competition, which damages expected retailer profits. By comparing our model with a fixed-price model, we are able to show that dynamic pricing is most beneficial when competition is not too intense.

With price precommitment (i.e., fixed price), Lippman and McCardle (1997), Mahajan and van Ryzin (2001), and Netessine and Rudi (2003) showed that competition consistently leads to overstock. Similarly, Netessine and Shumsky (2004) found that horizontal competition lowers the booking limit for lower-fare passengers. Without price precommitment, Kreps and Scheinkman (1983) showed that a capacity competition followed by a price competition leads to Cournot outcomes under efficient rationing of demand. Davidson and Deneckere (1986) demonstrated that mixed-strategy equilibria are often likely under the proportional rationing rule. Bikhchandani and Mamer (1993) derived a mixed-strategy equilibrium for an oligopolistic model and proved uniqueness for the duopoly case. All three models were static in pricing, and deterministic. Bernstein and Federgruen (2005, 2004) studied stochastic models in which inventory is set after price decisions. Because they assume that unmet demand is backlogged (and henceforth no demand substitution), the inventory problem was solved as a standard newsvendor model, and a reduced game in price was obtained. For other competitive dynamic pricing models that allow general demand functions and customer arrival processes, see Perakis and Sood (2005a, b). In this stream of research, our work can be viewed as extending the results of Kreps and Scheinkman (1983) for isoelastic demand to a more general environment with demand uncertainty and pricing flexibility.

Using a modified version of the taxonomy developed by Elmaghraby and Keskinocak (2003), our paper can be classified as C-S-NR-D-M (competition, single product, non-replenishment, dependent demand over time, and myopic customers), where we have added the first two dimensions of noncompetition versus competition and single product versus multiple products.

The remainder of this paper is organized as follows. In §2, we study the monopoly model. We consider the oligopoly case in §3. Conclusions and future research are given in §4. All proofs are presented in the appendix.
2. The Monopoly Model

We consider a retailer who purchases a quantity of a product at the beginning of a “season” and sells to price-sensitive customers over a finite time period. The problem faced by the retailer is to determine both an optimal order quantity and an optimal pricing policy.

We assume the following customer behavior model. Identical customers have a quasi-linear utility function
\[ u(x, e) = \frac{\alpha}{\alpha - 1} x^{(\alpha - 1)/\alpha} + sx + e \]
and face a budget constraint \( px + e \leq w \), where \( \alpha > 1 \), \( x \) is the consumption value, \( e \) is the numeraire, \( p \) is the price, and \( w \) is the wealth level. Hence, the customer’s demand function is \( x(p) = +\infty \) if \( p \leq s \) and \( x(p) = (p - s)^{-\alpha} \) if \( p > s \). Without loss of generality, we restrict price to the set \( p \in [s, +\infty) \). If \( s = 0 \), the demand function \( x(p) \) has a constant price elasticity coefficient \( \alpha \).

The “season” corresponds to the finite interval \([0, T]\). We let \( N(t) \) be the stochastic arrival rate of customers, which follows a geometric Brownian motion \( dN_t = N_t(\mu_dt + \sigma dB_t) \), or equivalently, \( N_t = e^{(\mu - (1/2)\sigma^2)t} + \int_0^t \sigma dB_s \), where \( \mu_t \) and \( \sigma_t \) are deterministic and time dependent. We define the history up to time \( t \) as \( H_t = \sigma\{N_s, 0 \leq s \leq t \} \) and \( h_t \) as a realization of \( H_t \). Let \( D(t, p(t)) = x(p(t))N(t) \) be the aggregate demand rate, where \( p(t) \) is the price at time \( t \) \((t \in [0, T])\). Define the aggregate inverse demand function as
\[ q(N(t), z) = s + \left( \frac{N(t)}{z} \right)^{1/\alpha} \]
We denote the quantity ordered by the retailer by \( y \) and the unit cost by \( c (>s) \). The retailer’s inventory at time \( t \) is \( y(t) = y - \int_0^t D(u, p(u)) \, du \). A pricing policy is given by \( p = p(t), t \in [0, T] \mid p(t) \) is a function of \( y(t) \) and \( h_t \). Denote \( \tau_{p,u,z} = \sup \{ t \in [u, T] \mid \int_u^t D(l, p(l)) \, dl \leq z \} \), where \( \zeta \geq 0 \), and set \( p(t) = +\infty \) when \( t > \tau_{p,u,z} \); that is, the demand process is “turned off” when the retailer runs out of inventory. Note that throughout this paper we will have their usual meanings because they are defined on nonempty sets.

Define the retailer’s expected revenue from \( t \) to \( T \) as a function of pricing policy \( p \), inventory level \( z \) at time \( t \), and history \( h_t \), by \( R(t, z, p, h_t) = E \left[ \int_t^T p(u) \cdot D(u, p(u)) \, du \mid h_t \right] \). Let \( R^*(t, z, p, h_t) = \sup_p R(t, z, p, h_t) \) be the maximal expected revenue from \( t \) to \( T \) given the history \( h_t \) and inventory \( z \) at time \( t \). Finally, define the expected profit at time zero as \( \Pi(y) = E[R^*(0, y, h_0)] - cy \) and let \( y^* \in \arg\max \Pi(y) \). Because the customer arrival rate process \( N_t \) is Markovian, we replace the full history \( h_t \) by the arrival rate \( n_t \) and rewrite expected revenue as \( R(t, z, p, n_t) = E \left[ \int_t^T p(u)D(u, p(u)) \, du \mid n_t \right] \) and \( R^*(t, z, p, n_t) = \sup_p R(t, z, p, n_t) \).

Define the elastic arrival forecast as
\[ a_t = \left( \int_t^T e^{\int_t^s \mu_r - ((a - 1)/2)\sigma_r^2 \, dr} \, ds \right)^{1/\alpha} n_t^{1/\alpha}, \quad t \in [0, T] \]
Note that given the customer arrival rate \( n_t \) at time \( t \), the expected number of total customer arrivals in \([t, T]\) is \( E[\int_t^T N_u \, du | n_t] = n_t \int_t^T e^{\int_t^s \mu_r dr} \, ds \), which is equal to \( a_t \) when \( \alpha = 1 \). Hence, \( a_t \) represents the expected number of customer arrivals between \( t \) and \( T \) with a calibration by the price elasticity \( \alpha \) and volatility \( \sigma \).

Theorem 1. The optimal expected revenue from \( t \) to \( T \) given the arrival rate \( n_t \) and inventory \( z_t \) at time \( t \) is \( R^*(t, z_t, n_t) = sz_t + a_t e^{1-1/\alpha} \), and the optimal pricing policy is \( p^* = \{ p^*(t) | p^*(t, z_t, n_t) = s + a_t e^{1-1/\alpha}, t \in [0, T] \} \).

Note that the optimal pricing policy \( p^* \) is Markovian. Theorem 1 gives us closed-form expressions for the optimal expected revenues and prices. Gallego and van Ryzin (1994) gave a closed-form optimal pricing policy for an exponential demand function of price and a Poisson arrival process. Because a Poisson process has independent increments, their optimal pricing policy is only a function of inventory, and hence is independent of the current demand information. In contrast, because we model arrivals with a geometric Brownian motion, which is Markovian, our pricing policy utilizes the current demand information \( N_t \).

We can characterize the optimal inventory levels in the following proposition.

Proposition 1. The optimal inventory level is given by
\[ y^*_t = y^*_0 \int_0^T e^{\int_0^s \mu_r - ((a - 1)/2)\sigma_r^2 \, dr} \, ds \]
which is deterministic and hits zero at time \( T \).

Because an isoelastic demand function implies that revenue is decreasing in price, the retailer has an incentive to set price as low as possible and sell all inventory. Moreover, she is enabled to do so because an isoelastic demand function takes values from 0 to +\( \infty \) as price decreases, and a customer arrival rate process, governed by a geometric Brownian motion, always provides a positive customer flow. Moreover, Proposition 1 implies that all stochasticity in customer arrivals is absorbed into the optimal pricing policy \( p^* \), which results in a deterministic inventory path. Note that the full value range (from 0 to +\( \infty \)) of isoelastic demand and the positivity of a geometric Brownian motion contribute to the results of Proposition 1. In contrast, a Poisson arrival process does not guarantee positive customer arrivals in a time interval. Some other demand functions, such as linear and exponential, imply that a retailer is not willing to price below the optimal price, which maximizes the concave revenue function and limits the range over which demand can vary. Hence, the optimal inventory path is often stochastic under other circumstances, such as a Poisson arrival process coupled with exponential demand (Gallego and van Ryzin 1994).

Figure 1(b) shows the optimal inventory trajectories \( y^*_t \) for the \( \mu_t \) values given in Figure 1(a), where \( y^*_0 = 1, \sigma = 2, \)}
we can characterize the dynamic behavior of the elastic arrival forecast, optimal prices, and consumer surplus in the following proposition.

**Proposition 2.** (1) \( \{a_t, H_t\} \) is a supermartingale;  
(2) \( \{p^*_t, H_t\} \) is a martingale; and  
(3) \( \{v^*_t, H_t\} \) is a submartingale.

Proposition 2 suggests that the retailer should keep prices “flat” ex ante. Note that if \( \sigma_t = 0 \) for \( t \in [0, T] \) (i.e., the customer arrival rate process is deterministic), \( p^*_t \) is constant on \( [0, T] \) and depends only on the realization of \( N_0 \). Hence, the martingale property of optimal prices can be considered a stochastic version of Proposition 2 in Gallego and van Ryzin (1994). Note that this property depends on the nature of customer arrivals. Instead, if customer arrivals follow a Poisson process instead of a geometric Brownian motion, optimal prices turn out to be a submartingale (see Xu and Hopp 2005b) because the retailer is not certain that there will be incoming customers, and hence tends to set price lower at the beginning of a season. In contrast, because a geometric Brownian motion always provides a positive customer flow, the retailer can count on future customer arrivals. Partially consistent with our results, Sapra and Jackson (2004) showed that the forecast process for future price evolves as a martingale for a general diffusion process, but the price process itself does not follow a martingale in their dynamic production and pricing model.

Finally, note that the expectation of consumer surplus is increasing. Hence, a strategic customer with full information will wait until the last minute to make a purchase. Of course, in many real-world settings customers cannot delay their purchase or behave strategically. For instance, customers may not have the information about demand and inventory possessed by the retailer, and hence cannot anticipate future prices. Furthermore, there may be nonprice characteristics (e.g., product availability) that are not captured in consumer surplus (see Xu and Hopp 2005b) because the retailer is not certain that there will be incoming customers, and hence tends to set price lower at the beginning of a season. In contrast, because a geometric Brownian motion always provides a positive customer flow, the retailer can count on future customer arrivals. Partially consistent with our results, Sapra and Jackson (2004) showed that the forecast process for future price evolves as a martingale for a general diffusion process, but the price process itself does not follow a martingale in their dynamic production and pricing model.

We can now complete the specification of an optimal policy for the monopoly case by characterizing the optimal inventory policy. To do this, note that by Theorem 1, \( R^* (0, y, h_0) = sy + a_0 y^{1-1/\alpha} \) and \( \Pi(y) = E[R^*(0, y, h_0)] - cy = E[a_0 y^{1-1/\alpha}] - (c-s)y \), where

\[
E[a_0] = E[N_0^{1/\alpha} \left\{ \int_0^T e^\mu u v^{-((\alpha-1)/2\alpha)\sigma^2} du \right\}^{1/\alpha}].
\]

We can optimize \( \Pi(y) \) to get

**Theorem 2.** The optimal initial inventory level is

\[
y^* = \left( 1 - \frac{1}{\alpha} \right) \frac{E[a_0]^\alpha}{(c-s)^{\alpha}}.
\]
and the optimal profit is
\[ \Pi^* = \Pi(y^*) = \frac{1}{\alpha} \left( 1 - \frac{1}{\alpha} \right) \frac{E[a_0]^\alpha}{(c - s)^{\alpha - 1}}. \]

A one-shot pricing model, in which a single price is set for the entire season before any demand information is revealed, has been widely studied (see, e.g., Petruzzi and Dada 1999 for a review). Monahan et al. (2004), in Proposition 5 and Corollary 1, compared the effect of dynamic pricing (recourse) with a one-shot pricing model, and found that pricing flexibility induces a higher initial inventory level and larger expected profit. We obtain the same conclusion in the following proposition.

**Proposition 3.** Assume that \( s = 0 \). Denote the optimal inventory and expected profit for one-shot pricing by \( \bar{y} \) and \( \bar{\Pi} \). Let \( \bar{V} = \max_{k \geq 0} V(k) \), where \( V(k) = k^{1/\alpha - 1}(k - \int_0^1 N, du)^{1/\alpha - 1} \). Then, \( \theta_j = y^*/\bar{y} = \Pi^*/\bar{\Pi} = (E[a_0]/\bar{V})^\alpha > 1 \), where \( \theta_j \) measures the value of pricing flexibility with inventory coordination.

Because the one-shot pricing model and the dynamic pricing model generate revenues \( y^{1-1/\alpha} \bar{V} \) and \( y^{1-1/\alpha} E[a_0] \), respectively, for fixed initial inventory \( y \), we may use \( \theta_p = E[a_0]/\bar{V} > 1 \) to measure the value of pricing flexibility without inventory coordination. Because \( \alpha > 1 \) and \( \theta_p > 1 \), it follows that \( \theta_j = \theta_p \alpha > \theta_p \); that is, pricing flexibility is more beneficial when coordinated with inventory procurement than when inventory is not taken into consideration. We further explore the value of pricing flexibility in the following example.

**Example 1.** Let \( T = 10 \), \( \alpha \in [1.5, 4] \), \( N_0 = 1 \), \( \mu_i = 0 \), and \( \sigma_i = \sigma \in [0.1, 3] \). We generate the contour graphs of \( \theta_j \) and \( \theta_p \) in Figure 2. Note that the value of pricing flexibility is asymptotically stable as the uncertainty of customer arrivals becomes large (i.e., \( \sigma \to +\infty \)), but is sensitive to price elasticity. In contrast, the value of pricing flexibility is insensitive to \( \alpha \), but is predominantly determined by \( \sigma \) if the customer arrival rate is relatively certain (i.e., \( \sigma \) is small).

Monahan et al. (2004, Figure 2) illustrated that the value of pricing flexibility with inventory coordination increases in \( \alpha \) and \( \sigma \). However, they only allow approximately 10 price changes, while we allow an infinite number of price changes. Hence, it is not surprising that their values of pricing flexibility are much lower than ours. As such, our results can be considered an upper bound on the value of pricing flexibility.

Moreover, it is interesting to observe that the value of pricing flexibility with inventory coordination (\( \theta_j \)) is increasing in price elasticity (\( \alpha \)), but the value of pricing flexibility without inventory coordination (\( \theta_p \)) is decreasing in \( \alpha \). Hence, as demand becomes more elastic, pricing flexibility becomes more valuable only if it is coordinated with the inventory decision; otherwise, elastic demand devalues pricing flexibility because it is difficult to adjust prices without losing sales.

![Figure 2.](image)

(a) The value of pricing flexibility with inventory coordination \( \theta_j \); (b) the value of pricing flexibility without inventory coordination \( \theta_p \).

### 3. The Oligopoly Model

In this section, we generalize the monopoly model of §2 into an oligopoly model, in which customers are represented by an atomic flow. That is, their instantaneous demand is infinitesimal and can be fulfilled by the retailers with the lowest market price and positive inventory.

To formulate a model, we assume that \( n \) retailers procure an amount of the product at a unit cost of \( c \). Let \( p_i(t) \) and \( y_i(t) \) be retailer \( i \)'s price and inventory level at time \( t \), where \( y_i(0) = y_i, i = 1, \ldots, n \), represent initial inventory levels. At time \( t \), define the price vector \( \mathbf{p}(t) = (p_1(t), \ldots, p_n(t)) \), inventory vector \( \mathbf{y}(t) = (y_1(t), \ldots, y_n(t)) \), and aggregate inventory level \( Y(t) = \sum_{i=1}^n y_i(t) \). Denote the set of low-cost retailers as \( I_1(\mathbf{p}(t)) = \{1 \leq i \leq n \mid p_i(t) \leq p_j(t), j \neq i\} \), the set of retailers with positive inventory as \( I_2(\mathbf{y}(t)) = \{1 \leq i \leq n \mid y_i(t) > 0\} \), and define \( I(\mathbf{p}(t), \mathbf{y}(t)) = I_1(\mathbf{p}(t)) \cap I_2(\mathbf{y}(t)) \). Hence, only retailers in \( I(\mathbf{p}(t), \mathbf{y}(t)) \) can make instantaneous sales at time \( t \). Denote \( P(\mathbf{p}(t), \mathbf{y}(t)) = p_i(t) \), where \( i \in I(\mathbf{p}(t), \mathbf{y}(t)) \), as the instantaneous sale price. Because customers are indifferent among retailers in \( I(\mathbf{p}(t), \mathbf{y}(t)) \), we assume the following splitting rule. Let \( \Psi(t) = (\psi_1(t), \ldots, \psi_n(t)) \) be an \( n \)-dimensional random process on \([0, T]\), which takes positive values and is independent with \( \{N(t), 0 \leq t \leq T\} \).

We assume that all retailers know the distribution of \( \Psi(t) \). Define \( \varphi_i(t, \mathbf{p}(t), \mathbf{y}(t)) = \varphi_i(t, \mathbf{p}(t), \mathbf{y}(t)), \ldots, \varphi_n(t, \mathbf{p}(t), \mathbf{y}(t)) \), where \( \varphi_j(t, \mathbf{p}(t), \mathbf{y}(t)) = 0 \) if \( i \notin I(\mathbf{p}(t), \mathbf{y}(t)) \).
and \( \varphi_i(t, \mathbf{p}, \mathbf{y}(t)) = \psi_i(t) / \sum_{j \in I} (p_j(t), \mathbf{y}(t)) \psi_j(t) \) if \( i \in I(p(t), \mathbf{y}(t)) \). The demand \( d(t, P(\mathbf{p}(t), \mathbf{y}(t))) \) is allocated to retailers in \( I(p(t), \mathbf{y}(t)) \) by \( \varphi_i(t, \mathbf{p}(t), \mathbf{y}(t)) \). Finally, this splitting rule implies that the inventory of retailer \( i \) is \( y_i(t) = y_i(0) - \int_0^t D(u, p_i(u)) \varphi_i(u, \mathbf{p}(u), \mathbf{y}(u)) \, du \) at time \( t \) and the aggregate inventory level is \( Y(t) = Y(0) - \int_0^t D(u, P(\mathbf{p}(u), \mathbf{y}(u))) \, du \).

To interpret the splitting rule \((\psi, \varphi)\), we consider the following example. Suppose that retailers are differentiated by some physical characteristics (e.g., store location and service quality) that provide time-dependent random utility \( U_i(t) \) for each retailer \( i = 1, \ldots, n \). If customers pick a store according to a logit model, then

\[
\psi_i(t) = \frac{e^{U_i(t)}}{\sum_{j=1}^{n} e^{U_j(t)}} \quad \text{and} \quad \varphi_i(t, \mathbf{p}, \mathbf{y}(t)) = \frac{e^{U_i(t)}}{\sum_{j \in I(p(t), \mathbf{y}(t))} e^{U_j(t)}},
\]

that is, customers choose the store with the lowest price, best fit, and positive inventory.

We now set up a two-stage game consisting of an inventory investment competition followed by dynamic pricing competition. We use weak perfect Bayesian equilibrium (WPBE) to characterize the outcome. Unlike the monopoly case, where we only assume the monopolist observes the customer arrival rate process, we need more restrictive information assumptions here for tractability. Hence, we assume that retailers know the initial aggregate inventory level \( Y(0) \). In a manner similar to that used by Dockner et al. (2000, §6.1), we define the information structure of the pricing subgame. That is, we assume at time \( t \) that retailers know the instantaneous sale price \( P(\mathbf{p}(t), \mathbf{y}(t)) \), so that every retailer can calculate the aggregate inventory level \( Y(t) \). Of course, retailer \( i \) also observes her own inventory level \( y_i(t) \) at time \( t \) and so processes public and private information given by \( \{Y(t), y_i(t), h_i\} \). For simplicity, we assume that retailers make price decisions based only on customer arrivals, the current inventory, and aggregate inventory levels. A pricing policy of retailer \( i \) is \( p_i = \{p_i(t), t \in [0, T]\} \), where \( p_i(t) \) is a function of \( Y(t), y_i(t), h_i(t) \) and we denote \( \mathbf{p}_i = (p_1, \ldots, p_n) \) and \( \mathbf{p}_- = (p_1, \ldots, p_i-1, p_{i+1}, \ldots, p_n) \). Note that the pricing game is with imperfect recall (see Piccione 1997) because price and inventory histories are ignored. Our equilibrium does not depend on this restriction, but it avoids the technical concerns of defining strategies on a functional space.

We define retailer \( i \)'s expected revenue from \( t \) to \( T \) given pricing policies \( \mathbf{p} \), inventory levels \( \mathbf{y}(t) = \mathbf{z}_i \), and history \( h_i \), as

\[
R_i(t, \mathbf{p}, \mathbf{z}_i, h_i) = E \left[ \int_t^T p_i(u) D(u, p_i(u)) \varphi_i(u, \mathbf{p}(u), \mathbf{y}(u)) \, du \right] | \mathbf{z}_i, h_i.
\]

Because retailer \( i \) does not know the inventory levels of the other retailers, she forms her belief \( \mathbb{E}_{i, t} \), which is a subjective multivariate distribution of competitor inventory levels. Given retailer \( i \)'s inventory \( y_i(t) = z_i, t \), and aggregate inventory level \( Y(t) = z_T \), at time \( t \), we define the expected revenue of retailer \( i \) as

\[
R_i(t, \mathbf{p}, \mathbf{z}_i, h_i) = \int_{A(Z, z_i)} R_i(t, \mathbf{p}, \mathbf{z}_i, h_i) \, d\mathbb{E}_{i, t}(Z, z_i),
\]

where \( Z_{-i,t} = (z_{1,t}, \ldots, z_{i-1,t}, z_{i+1,t}, \ldots, z_n, t) \) and

\[
A(Z, z_i) = \left\{ z_{-i,t} \left| \sum_{j \neq i} z_{j,t} = Z_t - z_{i,t} \right. \right\}.
\]

Let \( R^*_i(t, \mathbf{p}_i, z_i, h_i) = \sup_{\mathbf{p}_i} R_i(t, \mathbf{p}_i, z_i, h_i) \) be the maximal expected revenue from \( t \) to \( T \) given retailer \( i \)'s belief \( \mathbb{E}_{i, t} \) and the other retailers' pricing policies \( \mathbf{p}_{-i} \). Define the expected profit of retailer \( i \), \( \Pi_i(\mathbf{y}, \mathbf{p}, \mathbb{E}) = E[R_i^*(0, \mathbf{y}(0), Y_0, h_0)] - c_y \), where \( \mathbf{y} = (y_1, \ldots, y_n) \) are the initial inventory levels and \( Y_0 = \sum_{i=1}^{n} y_i \) is the aggregate inventory level at time \( 0 \).

We call \((\mathbf{y}^*, \mathbf{p}^*, \mathbb{E}^*)\) a WPBE if (1) the sequential rationality condition holds, that is, \( y_i^* \in \arg\max \{h_{-i}, \Pi_i(\mathbf{y}, \mathbf{p}, \mathbb{E}) \} \) and \( p_i^* \in \arg\max R_i^*(t, \mathbf{z}_i, h_i) \) for every \( t, z_i, h_i \); (2) belief \( \mathbb{E}^* \) is consistent, that is, it is derived from \((\mathbf{y}^*, \mathbf{p}^*)\) using Bayes’ rule whenever possible (see, e.g., Fudenberg and Tirole 1991 for details). Because individual inventory levels are not public information (and hence are unavailable to competitors), the only proper subgame is the whole game. The concept of WPBE is widely used for this type of dynamic games.

By Theorem 1, we know that the centralized optimal pricing policy is \( p_i^* = \{p_i^*(t) = s + a Z_t^{-\alpha} \}_{t} \), and the optimal expected revenue from \( t \) to \( T \) given history \( h_t \), is \( R_i^*(t, z_i, h_t) = z_T + a Z_t^{-\alpha} \), where \( Z_t \) is the aggregate inventory level at time \( t \). Pricing policy \( p_i^* \) is public, that is, \( p_i^* \) is a function of \( \{z_i, h_i\} \). We refer the reader to Fudenberg et al. (1994) for the detailed derivation. In the following theorem, we show that \( \mathbf{p}^* \) is a WPBE for the price game and \( R_i(t, \mathbf{p}_i^*, z_i, h_i) = R_i(\mathbf{p}_i^*, z_i, h_i) \) for any belief \( \mathbb{E}_{i, t} \) (i.e., (\( \mathbf{R}_i(t, \mathbf{p}_i^*, z_i, h_i) \) does not depend on \( z_{-i,t} \)).

**Theorem 3.** \( \mathbf{p}^* = (p_1^*, \ldots, p_n^*) \) is a WPBE for the price game under consistent belief \( \mathbb{E}^* \), where \( \mathbb{E}^* = (\mathbb{E}_{1, t}, \ldots, \mathbb{E}_{n, t}) \) and \( R_i(t, \mathbf{p}_i^*, z_i, h_i) = R_i(\mathbf{p}_i^*, z_i, h_i) = (s + a Z_t^{-\alpha}) z_i \).

Theorem 3 shows that all retailers are willing to behave cooperatively in a decentralized pricing system. Because the revenue function under constant demand elasticity is strictly decreasing in price, retailers want to sell all their inventory at a price that exactly clears the market. When demand is uncertain, this intuition still holds, because prices form a martingale in the centralized system and clear the market, and, due to the lack of holding costs or discounting, retailers do not have an incentive to sell their inventories earlier in the season.
Note that there are many WPBEs for the price competition. For instance, one retailer could price higher than \( p^*_i \) until the others run out of their inventories and then begin to price at \( p^*_i \). To put it rigorously, assume that the first \( n - 1 \) retailers play the centralized optimal pricing policy \( p_i = p^*_i \). Retailer \( n \) plays \( p_n(t) = p^*_i(t) + 1 \) as long as
\[
Z_t = \int_0^t e^{\mu_0 s - ((\alpha - 1)/2a) s^2} \, ds > \zeta_{n,i},
\]
otherwise, she plays \( p_n(t) = p^*_i(t) \). Obviously, this gives another WPBE with the same outcomes because everyone follows \( p^*_i \).

We now turn to the inventory competition. By Proposition 2, \( \Pi_1(y, p^*_i) = E[a_0]Y_0^{-1/n}y_i - (c - s)y_i \), where we drop the beliefs because \( p^*_i \) is public. With these we can demonstrate that the inventory replenishment competition has an equilibrium that leads to overstocking relative to the centralized solution.

**Theorem 4.** There is an equilibrium for the inventory replenishment competition, given by
\[
y^*_i = \frac{1}{n} \left( 1 - \frac{1}{n\alpha} \right) a E[a_0]^a (c - s)^{-a}, \quad i = 1, \ldots, n,
\]
which yields equilibrium profit
\[
\Pi^*_i = \frac{1}{n^2\alpha} \left( 1 - \frac{1}{n\alpha} \right)^{a-1} E[a_0]^a (c - s)^{-a+1}.
\]

Although we cannot guarantee that the inventory equilibrium is unique, we can show that an analogous discrete-time competition has a unique equilibrium. (See the online appendix at or.pubs.informs.org/companion.html for a “proof outline,” which reduces the question of uniqueness to a specific condition, and also gives the uniqueness proof for the discrete version of the game.)

We can compare the aggregate inventory and profit in the decentralized system with those for the centralized system, which by Theorem 2 are
\[
Y^*_d = \left( 1 - \frac{1}{\alpha} \right) a E[a_0]^a (c - s)^a
\]
and
\[
\Pi^*_d = \frac{1}{\alpha} \left( 1 - \frac{1}{\alpha} \right)^{a-1} E[a_0]^a (c - s)^{a-1}.
\]
by means of an overstock ratio
\[
\delta(n) = \frac{Y^*_d}{Y^*_c} = \left( \frac{\alpha - 1/n}{\alpha - 1} \right)^a > 1
\]
and profit ratio
\[
\frac{\Pi^*_d}{\Pi^*_c} = \frac{\alpha - 1}{n\alpha - 1} \left( \frac{\alpha - 1/n}{\alpha - 1} \right)^a < 1.
\]

Although these ratios do not depend on demand uncertainty, the differences are proportional to \( E[a_0]^a \). For any sample path, the aggregate inventory in the decentralized system is higher than that in the centralized system by a factor of
\[
\left( \frac{\alpha - 1/n}{\alpha - 1} \right)^a.
\]
(Note that in the decentralized system, every retailer’s inventory path is indeed stochastic if the splitting rule is random.) The sample path of the equilibrium price in the decentralized system is lower than the one in the centralized system (after shifting \( s \)) by a factor of
\[
\frac{\alpha - 1}{\alpha - 1/n}.
\]
Hence, overstocking hurts retailer profits by depressing prices.

As we would expect, the above competition penalty increases in the number of firms. As we show below, competition ultimately drives profits to zero.

**Proposition 4.** (1) \( Y^*_d \) is increasing in \( n \) and converges to \( a E[a_0]^a (c - s)^a \); (2) \( \Pi^*_d \) is decreasing in \( n \) and converges to 0.

With price precommitment, Mahajan and van Ryzin (2001, Theorem 4) and Lippman and McCardle (1997, Theorem 8) also found that aggregate inventory converges to a constant, and the aggregate profit declines to zero as the number of firms goes to infinity.

If there is no stochasticity in customer arrivals (i.e., \( \sigma_t = 0 \), and hence demand is deterministic), Theorems 3 and 4 duplicate the result of Kreps and Scheinkman (1983) for isoelastic demand; that is, a capacity competition followed by a static price competition leads to Cournot outcomes because a fixed pricing policy is optimal. However, in a stochastic competitive environment, one-shot pricing competition with capacity constraints does not guarantee a pure-strategy equilibrium. Similar to the Bertrand-Edgeworth paradox (see, e.g., Vives 1999, Chapter 5), this occurs because every retailer has an incentive to reduce price by a “penny,” so she can sell first and avoid all demand uncertainty ex ante. Replacing a fixed-price competition by a dynamic pricing competition is essential here to fully hedge demand uncertainty, remove the incentive to cut price by a “penny,” and guarantee the existence of a pure-strategy equilibrium. This problem is also avoidable by enforcing price precommitment (e.g., under price regulation) and focusing only on inventory competition (see, e.g., Lippman and McCardle 1997, Mahajan and van Ryzin 2001, and Netessine and Rudi 2003).

With price precommitment, Lippman and McCardle (1997, Theorem 7) showed that under a deterministic splitting rule and some technical conditions, the aggregate inventory level in the decentralized system is equal to that in the centralized system, as is the aggregate profit. Although making a price precommitment avoids overstocking, it forgoes the value of pricing flexibility, which can be
quite large, as shown in Figure 2. Hence, in the remainder of this section, we consider the question of when retailers should make a price precommitment.

We assume that retailers can commit to a price, which maximizes the profit of the centralized system. By making this price precommitment, retailers receive total profits of

$$\bar{\Pi} = \frac{1}{\alpha} \left(1 - \frac{1}{\alpha}\right)^{\alpha - 1} \frac{(\bar{V})^\alpha}{(c - s)^{\alpha - 1}},$$

where $\bar{V}$ is defined in §2. Define

$$\theta(n) = \frac{\Pi_2}{\Pi} = \frac{\alpha - 1}{n\alpha - 1} \left(\frac{\alpha - 1}{\alpha - 1}\right)^\alpha \theta,$$

as the value of pricing flexibility in a competitive environment. If $\theta(n) > 1$, the value of pricing flexibility compensates for the loss due to overstocking, and hence retailers should not make price precommitment. If $\theta(n) < 1$, then price precommitment is attractive. By Proposition 4, $\lim_{n \to \infty} \theta(n) = 0$. Hence, as competition becomes more intense, pricing flexibility becomes less attractive and static pricing eventually becomes preferred. This is consistent with the recent movement toward simpler fare structures in the U.S. airline industry as competition has intensified (Business Travel News 2005). We can define $\bar{n} = \sup\{n \geq 1 \mid \theta(n) \geq 1\}$ as the maximum degree of competition under which retailers should choose dynamic pricing instead of price precommitment.

We further examine the factors that influence the choice between fixed and dynamic pricing in the following example.

**Example 2.** We continue Example 1. Figure 3 shows the maximum degree of competition $\bar{n}$ that supports dynamic pricing and the overstock factor $\delta(\bar{n})$ computed using simulated approximation. Note that the maximum degree of competition is increasing in $\alpha$ because pricing flexibility with inventory coordination is generally more valuable when price elasticity is high (as shown in Example 1). In contrast, the overstock factor is generally decreasing in price elasticity because it is difficult to dump inventory in a price-sensitive market without hurting profit. Both $\bar{n}$ and $\delta(\bar{n})$ are sensitive (insensitive) to $\alpha (\sigma)$ when $\sigma > 1$, but insensitive (sensitive) to $\alpha (\sigma)$ when $\sigma < 1$.

![Figure 3](image)

4. Conclusions and Future Research

This paper studies a one-shot inventory replenishment problem with dynamic pricing. We assume that the customer arrival rate process is represented by a geometric Brownian motion and that customer demand is isoelastic. In the monopoly case, we find a closed-form optimal pricing policy that is a function of inventory and current demand information. Under this pricing policy, the inventory trajectory is deterministic, and all inventory is sold. We find that dynamic pricing substantially outperforms fixed pricing, particularly when coordinated with the inventory decision if demand is elastic. In the oligopoly case, we establish a WPBE for a game of inventory investment followed by dynamic pricing. We find that the pricing equilibrium is cooperative even in a noncooperative environment, but that inventory competition results in overstocking, and hence reduces profits. Finally, we examine the trade-off between dynamic pricing and price precommitment and find that flexible pricing is still beneficial, provided that competition is not too intense.

In this paper, we assume an exogenous customer arrival rate process. Because the customer surplus along the optimal price path is increasing, endogenization of the customer decision process (e.g., allowing customers to delay purchase for a lower price) may change the customer flow process. This would also cause the optimal pricing policy derived in this paper to no longer be optimal. Although there have been some recent attempts to include strategic customer behavior in revenue management (see, e.g., Aviv and Pazgal 2003, Tong and Dasu 2004, and Xu and Hopp 2005b), we believe that there are many more opportunities for research into this complex topic.

We also assume complete information about the distribution of customer arrivals on the part of the retailers. For many seasonal or fashion products, it is impossible to know before the season whether or not they will be hot sellers. Therefore, in practice, retailers gradually refine their information about demand parameters through real-time sales data and adjust their prices accordingly. In such
cases, demand learning plays an important role in determining price trends (see, e.g., Aviv and Pazgal 2002 and Xu and Hopp 2005a). How to optimally combine learning and pricing is still an open research question.

In this paper, retailers only need public signals to form a cooperative equilibrium in which individual beliefs do not matter. Although this cooperative pricing strategy greatly facilitates the analysis of a noncooperative inventory competition, it omits something of the nature of price competition. In cases where it is beneficial for retailers to use private signals, such as their own inventory level, the pricing game shifts from public monitoring to private monitoring, in which individual beliefs are fundamental. Private monitoring problems are much harder than public monitoring problems, and many are still open questions (see, e.g., Kandori 2002 for a review).

Finally, most dynamic pricing literature in the operations management field, including this paper, assumes maximization of revenue/profit, but dynamic pricing can serve other strategic goals, such as forming coalitions and delivering signals. For instance, pricing is an important mechanism with which airline companies achieve tacit collusion in multimarkets (see the Wall Street Journal 1990 for details). Omitting these nonprofit factors may lead to biased conclusions. Hence, these issues present research challenges to the incorporation of dynamic pricing into models of operations management.

**Appendix. Proofs**

**Proof of Theorem 1.** To maximize $R(t, z, p, n_t)$, it is easier to use the inverse demand variable $x$ as the control variable (instead of price) and rewrite the problem as

$$
0 = \max \left\{ xq(n_t, x) - xR^*_n(t, z, n_t) + R^*_n(t, z, n_t) n_t \mu_t + \frac{1}{2} R^*_n(t, z, n_t) n_t^2 \sigma_t^2 \right\}.
$$

Because $R^*_n(t, z, n_t) = s + (1 - 1/\alpha) a_t z_{t-1}^{-1/a} - x R^*_n(t, z, n_t) n_t = n_t^{1/\alpha} x - x R^*_n(t, z, n_t) n_t$, which gives $p^*(t, z, n_t) = q(n_t, x^t) = s + a_t z_{t-1}^{-1/a}$ and $x^t q(n_t, x^t) = x^t R^*_n(t, z, n_t) n_t = (1/\alpha) n_t a_t z_{t-1}^{-1/\alpha}$. Furthermore,

$$
R^*_n(t, z, n_t) = \frac{\partial a_t}{\partial n_t} z_{t-1}^{-1/\alpha} = \frac{1}{\alpha n_t} a_t z_{t-1}^{-1/\alpha},
$$

which implies

$$
R^*_n(t, z, n_t) n_t \mu_t = \frac{\mu_t}{\alpha} a_t z_{t-1}^{-1/\alpha}.
$$

Because

$$
R^*_n(t, z, n_t) = \frac{1 - \alpha}{\alpha^2 n_t^2} a_t z_{t-1}^{-1/\alpha},
$$

$$
\frac{1}{2} R^*_n(t, z, n_t) n_t^2 \sigma_t^2 = \frac{(1 - \alpha) \sigma_t^2}{2 \alpha^2} a_t z_{t-1}^{-1/\alpha}.
$$

Finally,

$$
R^*_n(t, z, n_t) = \frac{\partial a_t}{\partial n_t} z_{t-1}^{-1/\alpha} = -\frac{1}{\alpha} a_t z_{t-1}^{-1/\alpha} - \frac{1}{\alpha} a_t \left( \mu_t - \frac{\alpha - 1}{2} \sigma_t^2 \right) z_{t-1}^{-1/\alpha} = -\left( x^t q(n_t, x^t) - x^t R^*_n(t, z, n_t) \right)
$$

$$
- \frac{1}{2} R^*_n(t, z, n_t) n_t \mu_t - \frac{1}{2} R^*_n(t, z, n_t) n_t^2 \sigma_t^2.
$$

Hence, the HJB equation holds for $R^*(t, z, n_t) = s z_t + a_t z_{t-1}^{-1/\alpha}$ and $p^*(t, z, n_t) = s + a_t z_{t-1}^{-1/\alpha}$. By Fleming and Soner (1993), Theorem 8.1, there exists an optimal Markov control $p^*$, and the claim holds. □

**Proof of Proposition 1.** By Theorem 1, the optimal inventory level at time $t$ follows

$$
dy^*_t = -D(t, p^*(t, y^*_t, N_t)) dt = -N_t a_t^{-\alpha} y^*_t \sigma_t^2 dt.
$$

Hence,

$$
y^*_t = y_0 e^{-\int_0^t N_t a_t^{-\alpha} \sigma_t^2 du},
$$

$$
= y_0 e^{-\int_0^t \left( \int_0^t e^t \mu_t - (1 - (1/2) \alpha) \sigma_t^2 dx \right) du}.
$$

**Proof of Proposition 2.** Recall that

$$
a_t = \left\{ \int_0^T e^t \mu_t - (1 - (1/2) \alpha) \sigma_t^2 dt \right\}^{1/\alpha} N_t^{1/\alpha},
$$

where $N_t = \int_0^T e^t \mu_t dx + \int_0^T \mu_t dx + \int_0^T \sigma_t^2 dx du + \int_0^T \sigma_t^2 dx du = e^{-\int_0^T (2/\alpha) \sigma_t^2 dx} e^{-\int_0^T (1/\alpha) \sigma_t^2 dx} e^{-\int_0^T \sigma_t^2 dx} du + \int_0^T \sigma_t^2 dx du$.

It is easy to see that

$$
e^{-\int_0^T (2/\alpha) \sigma_t^2 dx} e^{-\int_0^T (1/\alpha) \sigma_t^2 dx} e^{-\int_0^T \sigma_t^2 dx} du + \int_0^T \sigma_t^2 dx du
$$

is a martingale. Hence,

$$
a_t = \left\{ \int_0^T e^t \mu_t - (1 - (1/2) \alpha) \sigma_t^2 dt \right\}^{1/\alpha}
$$

$$
\cdot N_t^{1/\alpha} e^{-\int_0^T \left( \mu_t - (1 - (1/2) \alpha) \sigma_t^2 \right) dx + \int_0^T \sigma_t^2 dx du + \int_0^T \sigma_t^2 dx du + \int_0^T \sigma_t^2 dx du,\ H_t
$$

where $N_t = \int_0^T e^t \mu_t dx + \int_0^T \mu_t dx + \int_0^T \sigma_t^2 dx du + \int_0^T \sigma_t^2 dx du = e^{-\int_0^T (2/\alpha) \sigma_t^2 dx} e^{-\int_0^T (1/\alpha) \sigma_t^2 dx} e^{-\int_0^T \sigma_t^2 dx} du + \int_0^T \sigma_t^2 dx du$.
which is a supermartingale because
\[
\left\{ \int_t^T e^{\int_0^l \mu_i - ((a-1)/2a_2^2 \sigma_i^2 \, dl)} d\mu \right\}^{1/a}
\]
is decreasing in \( t \).

Similarly,
\[
p_i^* = s + a_i(y_i^*)^{-1/a}
= s + (y_i^*)^{-1/a} \left( \int_0^T e^{\int_0^l \mu_i - ((a-1)/2a_2^2 \sigma_i^2 \, dl)} \right)^{1/a} \\
\cdot e^{-1/a \int_0^l \mu_i - ((a-1)/2a_2^2 \sigma_i^2 \, dl) \, dl} N_i^{1/a} e^{\int_0^l \sigma_i \, dB_d} d\mu \\
= s + (y_i^*)^{-1/a} \left( \int_0^T e^{\int_0^l \mu_i - ((a-1)/2a_2^2 \sigma_i^2 \, dl)} \right)^{1/a} \\
\cdot N_i^{1/a} e^{-((1/2) \int_0^l \sigma_i^2 \, dl) + \int_0^l \sigma_i \, dB_d},
\]
which implies Claim 2.

Finally,
\[
v_i^* = \frac{1}{\alpha - 1} (p_i^* - s)^{-1/a}
= \frac{1}{\alpha - 1} \left( y_i^* \right)^{(a-1)/a} \left( \int_0^T e^{\int_0^l \mu_i - ((a-1)/2a_2^2 \sigma_i^2 \, dl)} \right)^{(1-a)/a} \\
\cdot e^{-((1/2) \int_0^l \sigma_i^2 \, dl) + \int_0^l \sigma_i \, dB_d} N_i^{1/a} e^{\int_0^l \sigma_i \, dB_d}
= \frac{1}{\alpha - 1} \left( y_i^* \right)^{(a-1)/a} \left( \int_0^T e^{\int_0^l \mu_i - ((a-1)/2a_2^2 \sigma_i^2 \, dl)} \right)^{(1-a)/a} \\
\cdot N_i^{1/a} e^{-((1/2) \int_0^l \sigma_i^2 \, dl) + \int_0^l \sigma_i \, dB_d} e^{-((1/2) \int_0^l \sigma_i^2 \, dl) + \int_0^l \sigma_i \, dB_d},
\]
Because \( \alpha > 1 \), \( \{v_i^*, H_i\} \) is a submartingale. □

**Proof of Theorem 2.** It is easy to see that \( \Pi(y) \) is strictly concave in \( y \). By the first-order condition, the claim holds. □

**Proof of Proposition 3.** The expected profit for one-shot pricing is \( \Pi(y, p) = p(y - E[y - p^{-\alpha} \int_0^l N_i (ds)^{1/a}] - cy = y^{1-1/a} V(k) - cy \), where \( y \) is the initial inventory level, \( p \) is the price, and \( k = p^n y \). Let \( \tilde{k} \in \text{arg max } V(k) \). Then, the optimal inventory for one-shot pricing is
\[
y = \left( 1 - \frac{1}{\alpha} \right)^{\alpha} V \left( \tilde{k} \right) \left( \frac{1}{\alpha} \right) \left( \frac{1}{\alpha} \right)^{\alpha-1} \frac{\alpha-1}{c^{a-1}}.
\]
and the optimal profit is
\[
\Pi = \Pi \left( \tilde{y}, \left( \frac{\tilde{k}}{y} \right)^{1/a} \right) = \frac{1}{\alpha} \left( 1 - \frac{1}{\alpha} \right)^{\alpha-1} \left( \frac{\alpha}{c^{a-1}} \right).
\]
Because \( \Pi < \Pi^* \), the result follows directly. □

**Proof of Theorem 3.** Assume that Retailers 2 to \( n \) follow pricing policy \( p_i^* \). Furthermore, suppose that Retailer 1 is informed of the inventory levels of the other retailers and the splitting rule \( \Psi \). This cannot make Retailer 1 worse off and makes her pricing decision a standard stochastic control problem, similar to the one in Theorem 1. We claim that \( p_i^* \) is an optimal pricing policy for Retailer 1 under the conditions. Dropping subindices and representing Retailer 1’s expected revenue as \( R(t, p, z, Z_r) \), we focus on the inverse demand \( x \) (instead of price) as the control variable and rewrite the problem as
\[
\max_x \mathbb{E} \left[ \int_t^T f_i(x(u), Z(u)) q(N_u, x(du)) \mid n, z, Z_r \right]
\]
subject to
\[
dN_u = N_u (\mu_u \, du + \sigma_u \, dB_u),
\]
\[
dz = f_i(x(u), Z(u)) du,
\]
\[
dZ(u) = f_i(x(u), Z(u)) du,
\]
\[
z(u), Z(u) \geq 0,
\]
where \( z(u) \) and \( Z(u) \) are Retailer 1’s inventory level and aggregate inventory level, respectively, at time \( u \). For notational ease, we denote \( z(t) = z_r \) and \( Z(t) = Z_r \). If \( z(u) = Z(u) \), by Theorem 1, the claim holds. Hence, we assume that \( z(u) < Z(u) \) in the rest of the proof. Now note that \( f_i(x(u), Z(u)) = 0 \) if \( x(u) < a_u^{-\alpha} Z(u) n(u) \); \( f_i(x(u), Z(u)) = \varphi(u) x(u) \) if \( x(u) = a_u^{-\alpha} Z(u) n(u) \), and otherwise \( f_i(x(u), Z(u)) = x(u) \), where \( \varphi(u) \) is the splitting ratio if Retailer 1 offers the same price as the others. Similarly, \( f_i(x(u), Z(u)) = -a_u^{-\alpha} Z(u) n(u) \) if \( x \leq a_u^{-\alpha} Z(u) n(u) \), and otherwise \( f_i(x(u), Z(u)) = -x(u) \). The HJB equation (see Equation (III 8.1) in Fleming and Soner 1993) can be written as
\[
0 = \max_x \left\{ f_i(x, Z) q(n, x) - f_i(x, Z) R_i^c(t, z, Z_r, n) + f_x(x, Z) R_i^z(t, z, Z_r, n) \right. \]
\[
+ f_{Z_r}(x, Z) R_i^z(t, z, Z_r, n) + R_i^p(t, z, Z_r, n) \left( 1 - \frac{1}{\alpha} \right)^{\alpha} \left( \frac{1}{2} \right)^{\alpha} \frac{\alpha}{2\alpha} \frac{\alpha}{2\alpha} \right\}
\]
It is easy to check
\[
R_i^a(t, z, Z_r, n) n_i \mu_i = \frac{\mu_i}{\alpha} a_i Z_i^{-1/\alpha} z_i,
\]
\[
\frac{1}{2} R_i^a(t, z, Z_r, n) n_i \sigma_i^2 = \frac{(1 - \alpha)^{\alpha^2}}{2\alpha^2} a_i Z_i^{-1/\alpha} z_i^2, \quad \text{and}
\]
\[
R_i^c(t, z, Z_r, n) = -\frac{1}{\alpha} a_i^{-1} n_i Z_i^{-1/\alpha} z_i - \frac{1}{\alpha} \left( \mu_i - \frac{\alpha - 1}{2\alpha} \right) Z_i^{-1/\alpha} z_i.
\]
Summing these yields
\[
R_i^a(t, z, Z_r, n) n_i \mu_i + \frac{1}{2} R_i^a(t, z, Z_r, n) n_i \sigma_i^2 + R_i^c(t, z, Z_r, n) = -\frac{1}{\alpha} a_i^{-1} n_i Z_i^{-1/\alpha} z_i.
\]
It is also easy to check
\[ R^*_z(t, z_t, Z_t, n_t) = s + a_t Z_t^{-\alpha}, \]
\[ R^*_z(t, z_t, Z_t, n_t) = -\frac{1}{a_t} Z_t^{\frac{1}{\alpha}} z_t \]
and
\[ f_1(x, Z)q(n, x) - f_1(x, Z)R^*_z(t, z_t, Z_t, n_t) \]
\[ + f_2(x, Z)R^*_z(t, z_t, Z_t, n_t) = \frac{1}{\alpha} a_t^{\frac{1}{\alpha}} n_t Z_t^{-\frac{1}{\alpha}} z_t \]
if \( x \leq a_t^{-\alpha} Z_t n_t \),
and otherwise
\[ f_1(x, Z)q(n, x) - f_1(x, Z)R^*_z(t, z_t, Z_t, n_t) \]
\[ + f_2(x, Z)R^*_z(t, z_t, Z_t, n_t) = x^{1-\alpha} n_t^{\frac{1}{\alpha}} - x \left( a_t Z_t^{\frac{1}{\alpha}} - a_t^{\frac{1}{\alpha}} Z_t^{-\frac{1}{\alpha}} z_t \right), \]
which has derivative
\[ \left( 1 - \frac{1}{\alpha} \right) x^{1-\alpha} n_t^{\frac{1}{\alpha}} - a_t Z_t^{\frac{1}{\alpha}} a_t^{\frac{1}{\alpha}} Z_t^{-\frac{1}{\alpha}} z_t \]
\[ \leq \left( 1 - \frac{1}{\alpha} \right) a_t Z_t^{\frac{1}{\alpha}} a_t^{\frac{1}{\alpha}} Z_t^{-\frac{1}{\alpha}} z_t \]
\[ = a_t^{\frac{1}{\alpha}} Z_t^{\frac{1}{\alpha}} (z_t - Z_t) \leq 0. \]

Hence, the maximum is obtained by \( x^* = n_t a_t^{-\alpha} Z_t \), which gives
\[ p^*(t, z_t, Z_t, n_t) = q(n, x^*) = s + a_t^{-\alpha} Z_t \]
and
\[ f_1(x^*, Z)q(n, x^*) - f_1(x^*, Z)R^*_z(t, z_t, Z_t, n_t) \]
\[ + f_2(x^*, Z)R^*_z(t, z_t, Z_t, n_t) = \frac{1}{\alpha} n_t a_t^{\frac{1}{\alpha}} Z_t^{-\frac{1}{\alpha}} z_t. \]

Hence, the HJB equation holds for \( R^*(t, z_t, Z_t, n_t) = s z_t + a_t Z_t^{-\alpha} z_t \) and \( p^*_t \). By Fleming and Soner (1993, Theorem 8.1), the claim holds. \( \Box \)

**Proof of Theorem 4.** For \( i = 1, \ldots, n \),
\[ \frac{\partial \Pi_i(y, p^*_t)}{\partial y_i} = E[a_0] Y_0^{\frac{1}{\alpha} - 1} - \frac{1}{\alpha} E[a_0] Y_0^{\frac{1}{\alpha} - 1} y_i - (c - s) \]
and
\[ \frac{\partial^2 \Pi_i(y, p^*_t)}{\partial^2 y_i} = -\frac{1}{\alpha} E[a_0] Y_0^{\frac{1}{\alpha} - 1} \left( 2 - \left( \frac{1}{\alpha} + 1 \right) \frac{y_i}{Y_0} \right) < 0 \]
because \( \alpha > 1 \). Hence, \( \Pi_i(y, p^*_t) \) is concave in \( y_i \). By Fudenberg and Tirole (1991, Theorem 1.2), there exists a pure-strategy equilibrium for the inventory competition. Setting \( \partial \Pi_i(y, p^*_t)/\partial y_i = 0 \),
\[ E[a_0] \left( 1 - \frac{y_i}{\alpha Y_0} \right) = (c - s) Y_0^{\frac{1}{\alpha}}, \quad i = 1, \ldots, n. \]

Hence, the equilibrium is symmetric for \( p^*_t \). Letting \( y_1 = \cdots = y_n \) and solving the first-order condition, we obtain
\[ y_i^* = \frac{1}{n} \left( 1 - \frac{1}{\alpha n} \right)^{\alpha} E[a_0]^\alpha \left( \frac{c - s}{Y_0} \right)^{\alpha}, \quad i = 1, \ldots, n. \]

Plugging \( y_i^* \) into \( \Pi_i^*(y^*, p^*_t) \), we obtain \( \Pi_i^* \). \( \Box \)

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**References**


