A Static Approximation for Dynamic Demand Substitution with Applications in a Competitive Market

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We propose a static approximation of dynamic demand substitution behavior based on a fluid network model and a service-inventory mapping. This approximation greatly enhances our ability to analyze the interdependent inventory/service, price, and product assortment decisions in noncompetitive and competitive scenarios with demand substitution. We demonstrate that the approximation is well behaved and then apply it to two previously intractable applications. First, we study a price and service competition between single-product retailers. After establishing a unique pure-strategy Nash equilibrium, we find that competition results in lower price, higher demand, and a higher level of inventory. We also observe that the aggregate profit and inventory level increase to positive constants as the number of retailers goes to infinity. Second, we study a duopolistic competition on price, service, and product assortment. We establish a pure-strategy Nash equilibrium for the product assortment competition and identify a condition for uniqueness. We find that competition on both price and product assortment results in lower price and less variety for each competitor, but the total number of products and the aggregate inventory level in a duopoly market are both likely to be higher than in a monopolistic market.

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1. Introduction and Literature Review

Avoiding stockouts is a central concern of inventory management because running out of stock negatively affects the revenue of retailers and manufacturers and can erode customer loyalty to stores and brands. However, despite the large investment in information systems and inventory-tracking technology that has taken place over the past two decades, the worldwide out-of-stock rate remains surprisingly high 8.3% (Gruen et al. 2002). Stockouts induce some customers to change their purchase decisions. Gruen et al. (2002) surveyed more than 71,000 customers worldwide and found that 26% of customers encountering a stockout situation substitute brand, 19% substitute another product of the same brand, 31% buy the same product at another store, and 24% choose to delay or cancel their purchase.

Because of its prevalence, customer demand substitution is an important consideration for retailer inventory management, pricing, and category management strategies. Unfortunately, it is extremely difficult to accurately model demand substitution behavior, and hence the impact of stockouts on inventory replenishment, price competition, and product assortment is not well understood. In this paper, we provide an approximation of customer demand substitution behavior that greatly facilitates the embedding of stockout considerations in inventory, pricing, and product assortment models.

Demand substitution arises in two basic forms: firm-driven substitution and customer-driven substitution. Firm-driven substitution usually involves vertically differentiated products/components and often occurs in multiproduct manufacturing systems (see, e.g., Bassok et al. 1999 and Bitran and Dasu 1992). For example, a manufacturer may fill demand for a low-quality product with a high-quality product without charging a higher price. This substitution does not generate customer complaints when all customers agree with a ranking of product quality (e.g., 1G memory chips are better than 512M memory chips). In contrast to firm-driven substitution, customer-driven substitution involves a compensation scheme (e.g., charging a higher price) when a customer switches from one product to another, and hence does not require a monotonic ranking of products by heterogeneous customer perceptions. For example, a customer who purchases the higher-priced product when the lower-priced one is out of stock is practicing customer-driven substitution. Another example occurs when heterogeneous customers select from a horizontally differentiated product line. Because a seller has no
information of individual customer tastes, she cannot make a decision for customers. Customer-driven substitution governed by customers’ self-interested behavior is prevalent in retail settings. As such, it is more relevant to the inventory, pricing, and product assortment problems that motivated this work than is firm-driven substitution. Hence, this paper focuses on customer-driven substitution.

There are two major approaches to modeling customer-driven substitution. The first assumes that customer purchases are rationed according to a proportion rule when stockouts occur (see, e.g., Lippman and McCardle 1997, Netessine and Rudi 2003, Parlar 1988, and Smith and Agrawal 2000). The second approach explicitly models customer decision processes; that is, by assuming sequentially incoming customers carry a utility vector and purchase the available product (if one exists) that maximizes their utility (see, e.g., Mahajan and van Ryzin 2001a, b). This dynamic substitution approach is considerably more complex than the proportional rationing approach because it involves a dynamic sample-path model. Nevertheless, it has strong practical appeal because it is more natural to endogenize price and product quality decisions into the second approach than the first. Finally, we notice that there are some other approaches. For example, Lippman and McCardle (1997) studied a scenario under which all excess demand is randomly allocated to one single firm.

The technical difficulty inherent in dynamic substitution models arises due to (a) the difficulty of documenting each inventory sample path, and (b) the possibility that the profit function may not be quasi-concave in inventory levels (Mahajan and van Ryzin 2001a). Although Mahajan and van Ryzin (2001a) proposed a fluid model as an approximation, it does not fully resolve these technical difficulties. In this paper, we further simplify the dynamic fluid approximation into a static approximation, which vastly improves modeling tractability and facilitates inclusion of other decision variables, such as price and product assortment. To maintain simplicity, we assume that customer demand follows an attraction model (see, e.g., Basuroy 1997). Attraction models are widely used in the marketing literature for modeling customer choices and include the logit model as a special case. However, because our approximation relies on the IIA property (independence of irrelevant alternatives, which implies the ratio of the probability of choosing one product to the probability of choosing another is independent of the presence of the remaining products; see, e.g., Anderson et al. 1992) of attraction models, there is no direct generalization to other customer choice models that do not have the IIA property, such as those used by Mahajan and van Ryzin (2001a) and Smith and Agrawal (2000).

In our static approximation, we estimate the effective demand for a set of products with demand substitution by assigning each product a service rate and representing the demand process as a balanced network in which products are nodes and customer demands are flows. Then, we construct a one-to-one mapping between service rates and inventory levels by adopting a service rate function similar to that used in Deneckere and Peck (1995) and Dana (2001). Due to the demand substitution effect, the resulting effective demand function depends on inventory levels as well as other factors, such as prices and product quality. For example, the effective demand for a product increases in its inventory level but decreases in the inventory levels of other products. In addition to having these intuitive qualitative properties, we show that our static model yields a reasonably accurate quantitative approximation of the mean of the effective demands in the dynamic model of Mahajan and van Ryzin (2001a) when the coefficient of variation of the number of total customer arrivals is low (e.g., Poisson arrivals with high mean).

To illustrate its usefulness, we apply our static approximation to the analysis of two competitive scenarios, where the demand model is assumed to be logit. In the first application, we study a price and service competition among symmetric retailers, each of whom offers a single product. Focusing on service as a competitive dimension in operations management is realistic because, as reported in Chain Store Age (2002), customer service level is ranked as the number-one metric used for inventory management. The resulting game is similar to the one studied in Bernstein and Federgruen (2004b), in which they assumed that demand is a function of prices and service rates, and recovered inventory levels that guarantee precommitted service rates by means of a base-stock policy. In contrast, our model does not take effective demand as given, but instead derives it from an approximation of customer demand substitution behavior and calculates inventory levels by means of a service-inventory mapping. Hence, in our price and service game, we assume that all retailers use the static approximation.

With this demand model, we are able to establish a unique pure-strategy equilibrium for the oligopolistic price and service game. The equilibrium service rate is determined by the demand uncertainty and unit product cost and is independent of prices and other parameters. This is an implication of the logit demand function, which was also observed by Bernstein and Federgruen (2004b). We calculate the equilibrium inventory levels by using the service-inventory mapping and find that the aggregate inventory level in a decentralized system is higher than in a centralized system because competition lowers price and hence increases demand. Similar overstock phenomena have been observed in Lippman and McCardle (1997), Mahajan and van Ryzin (2001a), and Netessine and Rudi (2003). However, these previous studies took price as exogenous, and therefore the driver of overstock was inventory competition rather than pricing.

Note that a pure-strategy equilibrium in an inventory and price competition may not exist, and that existence is critically dependent on the demand-rationing rule adopted (see, e.g., Kreps and Scheinkman 1983 and Davidson and
Deneckere (1986). Zhao and Atkins (2005) identified conditions that guarantee existence of a pure-strategy equilibrium in an inventory and price competition game, but their demand-rationing rule is independent of prices. In contrast, our price and service competition with the service-inventory mapping provides us with a simple measure of equilibrium inventory levels. However, because our game is an approximation of the price and inventory game with demand substitution, our equilibrium predictions are only suggestive for the original game. Moreover, our price and service game requires a strong rationality of retailers, who must be able to anticipate the equilibrium service rates of the others and can adjust their inventory levels to implement the equilibrium service rates. Because service quality is often not directly implementable, the full rationality assumption of agents is needed for service competition, in which service quality is a decision variable. (See another example in Bernstein and Federgruen 2004a.)

For the oligopolistic service and price game, we also find that both aggregate equilibrium profit and aggregate equilibrium inventory level increase to positive constants as the number of retailers goes to infinity. In contrast, under an assumption of exogenous prices, Theorem 8 of Lippman and McCardle (1997) and Theorem 4 of Mahajan and van Ryzin (2001b) showed that increasing the number of retailers may cause aggregate equilibrium inventory level to converge to a constant, but cause aggregate equilibrium profit to drop to zero.

In the second application, we study an asymmetric duopolistic competition in price, service, and product assortment. We model the competition as a sequential game, consisting of an assortment subgame in the first stage followed by a price and service subgame in the second stage. In the assortment subgame, the competitors choose their product lines.

This formulation leads to the conclusion that every producer should offer an assortment of products with the largest quality markup indices, which are functions of customer perception of product quality, unit product cost, and demand uncertainty. Similar monotonic assortment results were shown in Aydin and Ryan (2000) (for the logit model, Hopp and Xu (2005) (for the Bayesian logit model), and van Ryzin and Mahajan (1999) (for the logit model and the trend-following population model). The first two models include price decisions but exclude inventory decisions. The third model includes inventory decisions but takes price as given and does not consider demand substitution. Mahajan and van Ryzin (2001a) extended van Ryzin and Mahajan (1999) to include demand substitution by a sample-path approach, but still took price as exogenous. Via numerical experiments, they found that considering demand substitution leads to more products being stocked. Singh et al. (2005) extended van Ryzin and Mahajan (1999) to include supply chain coordination. Using a continuous approximation, they found a similar monotonic assortment result holds in a supply chain. Maddah and Bish (2007) and Bish and Maddah (2004) studied a joint pricing, assortment, and inventory decision model for logit demand (still without demand substitution). They used numerical experiments to show that a monotonic assortment is nearly optimal and concluded that inventory cost tends to limit product variety. Smith and Agrawal (2000) found that demand substitution can substantially affect product assortment, inventory decisions, and profit. They also observed that stocking the most popular items may not be optimal. Gaur and Honhon (2006) used a location choice model and found that demand substitution affects product positions and offering the most popular items may be suboptimal. All of the above models were for monopolistic settings.

Although the operations management literature on product assortment and pricing problems has largely ignored competition, some economics papers have considered this issue (for reviews see Lancaster 1990, Kaul and Rao 1995, and Ramdas 2003). In a paper related to our work, Anderson and de Palma (1992) studied a symmetric oligopoly model, in which each producer competes on price and product line length. They assumed nested-logit demand, linear variable cost (i.e., no inventory), and a linear fixed cost of product line length, and established the uniqueness of a pure-strategy equilibrium for the product line competition. We also identify a condition for the uniqueness of a pure-strategy equilibrium for our assortment subgame, which depends on the shape of the product line cost function. Shugan (1989) studied price and assortment competition in a triopoly, with a constant elasticity demand function, and identified factors (e.g., competition and price sensitivity) that discourage or encourage larger product assortment. We find that the total number of products in a competitive market tends to be larger than in a monopolistic market, although both producers offer less variety than a monopolist does to lessen price competition.

Cachon and Kök (2007) used a nested multinomial logit model to study an assortment planning problem in category management. They found that decentralized category management provides less variety than centralized management does. They suggested a basket profit approach under a decentralized system, which achieves nearly the optimal variety and profit.

In summary, our demand substitution approximation enables us to generate a framework for considering, in an integrated manner, four factors: product assortment, price, inventory/service, and competition. In the remainder of the paper, we develop this framework as follows. We introduce the static approximation and study its analytical and numerical properties in §2. We apply the approximation to a price and service competition problem in §3 and to a product assortment problem in §4. We discuss conclusions and future research directions in §5. All proofs can be found in the online appendix. An electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/.
2. The Static Approximation of Dynamic Demand Substitution

We study a customer choice model similar to the one used in Mahajan and van Ryzin (2001a, b). Specifically, we consider a market with $L$ products, indexed by $\mathcal{L} = \{1, \ldots, L\}$, and assume that customers have a no-purchase option, indexed by zero. The market is open during a finite time interval $[0, T]$, called a “season,” and begins with initial inventory levels $y_i$ of products $i = 1, \ldots, L$. We assume that there is no inventory replenishment during the season, so some products may become unavailable during the season. We denote $\chi_i(t)$ as the indicator of the availability of product $i$ at time $t$; that is, $\chi_i(t) = 1$ if $y_i(t) > 0$; otherwise $\chi_i(t) = 0$, where $y_i(t)$ is the inventory level of product $i$ at time $t$ and $y_i(0) = y_i$. Hence, if a customer picks product $i$ at time $t$, and the product is in stock, the inventory level of product $i$ drops by one, i.e., $y_i(t+) = y_i(t) - 1$, and $\chi_i$ drops to zero from one when product $i$ stocks out (as illustrated in Figure 1). We let $\mathcal{S}(t) = \{i \in \mathcal{L} | \chi_i(t) = 1\}$ be the set of available products at time $t$. It is obvious that $\mathcal{S}(0) = \mathcal{L}$ and $\mathcal{S}(t) \subseteq \mathcal{S}(s)$, where $t > s$.

We assume that customer arrivals follow an exogenous stochastic process (e.g., a Poisson process) that is independent of the set of available products, and customer choice follows an attraction model. Specifically, at time $t$ we assume that a customer selects product $i$ from the set of products that are still available with probability $P_{i|\mathcal{S}(t)} = r_i/(r_0 + \sum_{j \in \mathcal{S}(t)} r_j)$, where $\{r_j\}_{j=0}^L$ are assumed to be independent of the product availability and time. Hence, $r_i$ is the attraction factor for product $i$, which may depend on the price and quality of product $i$. These assumptions are similar to those used in Smith and Agrawal (2000).

Given the product availability processes $\{\chi_i(t), t \in [0, T]\}_{i=1}^L$, we can calculate the expected effective demand for product $i$ as

$$E\left[ \int_0^T \frac{r_i \chi_i(t)}{r_0 + \sum_{j=1}^L r_j \chi_j(t)} dN(t) \right],$$

where $N(t)$ is the exogenous customer arrival process. We denote $c_i$ as the unit cost of product $i$, $p_i$ as the price, and $v_i$ as the salvage value and assume that $p_i > c_i > v_i$. Then, the expected profit for product $i$ is

$$p_i E\left[ \int_0^T \frac{r_i \chi_i(t)}{r_0 + \sum_{j=1}^L r_j \chi_j(t)} dN(t) \right]$$

$$+ v_i \{ y_i - E\left[ \int_0^T \frac{r_i \chi_i(t)}{r_0 + \sum_{j=1}^L r_j \chi_j(t)} dN(t) \right] - c_i y_i \}$$

$$= (p_i - v_i) E\left[ \int_0^T \frac{r_i \chi_i(t)}{r_0 + \sum_{j=1}^L r_j \chi_j(t)} dN(t) \right] - (c_i - v_i) y_i.$$

Hence, without loss of generality, we assume that $v_i = 0$, i.e., product $i$ has zero salvage value.

Due to the demand substitution effect, the product availability process $\chi_i(t)$ of product $i$ is jointly determined by inventory levels of all products, which results in a major technical difficulty. Mahajan and van Ryzin (2001a, b) applied a sample-path approach to model the inventory processes $\{y_i(t), t \in [0, T]\}_{i=1}^L$ and thereby generate realized demand and profit. In contrast to their approach, we ignore the time structure and propose a much simpler and more tractable static approximation of effective demand and product profit. The approximation procedure that we will develop in this section involves three major steps:

1. Replace the stochastic process $\chi_i(t)$ by a constant service rate;
2. Assume that the total number of customer arrivals $N(T)$ is a continuous random variable;
3. Approximate the customer demand substitution process by a fluid network with a memoryless flow property.

First, instead of counting the stochastic product availability process $\chi_i(t)$, we replace the stochastic process $\chi_i(t)$ by constant service rate $s_i$ (as shown in Figure 1). Hence, the expected effective demand for product $i$ can be approximated by $E \{ \int_0^T \frac{r_i s_i}{r_0 + \sum_{j=1}^L r_j s_j} \} = E \{ \int_0^T \frac{r_i \chi_i(t)}{r_0 + \sum_{j=1}^L r_j \chi_j(t)} dN(t) \}$, where $N = N(T)$. Next, we will introduce a service-inventory mapping, which determines the service rate $s_i$ given initial inventory levels of all products $\{y_i\}_{i=1}^L$. This critical step of our static approximation is based on a fluid network, which we describe below.

To begin with, it is easy to see that the customer choice probabilities $\{P_{i|\mathcal{S}(t)}\}_{i \in \mathcal{S}(t)}$ have the IIA property. This property enables us to interpret $\{P_{i|\mathcal{S}(t)}\}_{i \in \mathcal{S}(t)}$ as generated by a sequential substitution process, in which a customer is not informed of the set of available products $\mathcal{S}(t)$ and instead learns about it by substituting another product if an out-of-stock product is chosen. We make this explicit as follows.

Proposition 1. At time $t$, the choice probabilities $\{P_{i|\mathcal{S}(t)}\}_{i \in \mathcal{S}(t)}$ can be generated by a sequential substitution process as follows:

S0: Let $k = 1$ and $\mathcal{S}_k = \mathcal{L}$.
S1: A customer chooses product $j$ from $\mathcal{S}_k$ with probability $P_{i|\mathcal{S}_k}$, $j \in \mathcal{S}_k$.
S2: If product $j$ is available, stop; the customer gets product $j$. Otherwise, if product $j$ is not available (i.e., $j \not\in \mathcal{S}(t)$), then set $\mathcal{S}_{k+1} = \mathcal{S}_k - \{j\}$, increment $k = k + 1$, and go back to S1.
An illustration of the sequential substitution process.

Figure 2. An illustration of the sequential substitution process.

Figure 2 illustrates this sequential substitution process for the situation where $\mathcal{L} = \{1, \ldots, 6\}$ and $\mathcal{B}(t) = \{1, 3, 6\}$. Every product is represented as a node (solid square for an available product and dashed square for a stock-out product). Figure 2 includes three different substitution paths governed by the sequential substitution process in Proposition 1. If the customer picks product 1 (solid arrow), her demand is immediately satisfied. If, instead, she initially chooses product 5 (dashed dot arrow), then her demand is not met and so she tries another product. She may, for example, switch to product 6 with probability $P_{6|5} = 0.5$. In addition to switching products, the customer may choose to leave after failing to find a preferred product, possibly after multiple tries, as illustrated by the path $4 \rightarrow 2 \rightarrow 0$ (dashed arrows) in Figure 2.

Proposition 1 enables us to study the demand substitution problem as a network whose nodes (but only some of the arcs) are shown in Figure 2. The source node represents customer arrivals and each of the other nodes represents either a product or no-purchase option. Demand substitution is modeled as a flow along an arc (e.g., a flow from node $i$ to node $j$ represents a switch of unmet demand for product $i$ to demand for product $j$). The inflow at node $i$ is satisfied as long as there is positive inventory (of product $i$) at node $i$, which implies no outflow at node $i$ (i.e., node $i$ is a sink). Because initial inventories are finite, the set of available products becomes smaller as the season progresses, and hence the flow structure of the network also changes over time (i.e., node $i$ may change from a sink to a transient node). This makes it complicated to keep track of the substitution process, and hence very difficult to incorporate substitution behavior into other decision scenarios. To develop a simple approximation of the impact of substitution on inventory and demand, we ignore the dynamic nature of the set of available products. To do this we first introduce a service rate that measures the ratio of unmet demand to total demand.

**Definition 1.** For inventory level $y$, the service rate is defined as $f(y) = E[\min(y, N)/N]$, where $N$ represents the total number of customer arrivals, i.e., $N = N(T)$.

A similar service rate function, $\frac{E[\min(y, N)]}{E[N]}$, was used previously by Dana (2001) and Deneckere and Peck (1995). In their models, customers form rational expectation of service rates, which leads to equilibrium service rates being determined by incentive compatibility constraints, initial capacity, and the service-rate function. We will see how the static network approximation leads us to the service-rate function $f(y)$, which is analytically tractable for the applications in §§3 and 4. However, note that there may exist scenarios under which the service-rate function $E[\min(y, N)]/E[N]$ is a better alternative than $f(y)$.

In the rest of this paper, we assume that $N$ is a continuous random variable with density $h(n) > 0 \; \forall n \in (0, +\infty)$. We can characterize the service-rate function $f$ as follows:

**Proposition 2.** (1) $f(y)$ is differentiable and strictly increasing in $y$; (2) $f(y)$ is concave; (3) $f^{-1}(s)/s$ is strictly increasing in $s \in (0, 1)$, where $f^{-1}$ is the inverse of $f$.

We can use this service-rate function to avoid the complexities that result from dynamic inventory levels by approximating the dynamic network with a static network in which each node is assigned a fixed service rate. With this, we can calculate inflow and outflow rates that are consistent with the conservation law and determine the equilibrium service rates that result from the initial inventory and the service-rate function $f$. Finally, we can estimate effective demand, market share, and revenue of every product by using the inflow and outflow rates and the equilibrium service rates.

To make this procedure explicit, recall that $N$ is the total number of customer arrivals during the season. The static network is illustrated in Figure 3. Let $I_i$ be the inflow amount and $O_i$ be the outflow amount at node $i$. We assume that $100s_i\%$ of the inflow is absorbed at node $i$ (i.e., $100s_i\%$). Hence, $O_i = I_i(1-s_i)$ represents unmet demand. To generate a very simple approximation of the demand substitution effect, we ignore the source of the outflow and redirect $O_i$ back to every node according to the choice probabilities $P_{ij}$ (i.e., unlike the flows with the memory of their origins in Figure 2, flows are memoryless in the static network, as illustrated in Figure 3). The memoryless flow assumption is similar to Assumption 5* in Smith and Agrawal (2000), but Smith and Agrawal (2000) allowed only one attempted substitution if the customer’s best choice is out of stock. This simplification results in the tractability of the following analysis.
Following this assumption, total inflow \( I_i = (N + \sum_{j=1}^{L} O_j)P_{i|x} \), which implies \( I = N + \sum_{j=1}^{L} O_j = I_i/P_{i|x} \), \( i \in \mathcal{L} \). To ensure that the network follows the conservation law, \( I = N + I \sum_{j=1}^{L} P_{i|x}(1 - s_i) \) requires \( I = N/(P_{0|x} + \sum_{j=1}^{L} P_{i|x}s_j) \). Hence, the inflow at node \( i \),
\[
I_i = \frac{NP_{i|x}}{P_{0|x} + \sum_{j=1}^{L} P_{i|x}s_j} = \frac{N_{ri}}{r_0 + \sum_{j=1}^{L} r_j s_j}, \quad i \in \mathcal{L}.
\]

Given initial inventory \( y_i \) of product \( i \) and inflow \( I_i \) at node \( i \), the service rate (in an equilibrium sense) is given by \( s_i = E[\min(y_i/I_i, 1)] = E[\min(y_i/(r_0 + \sum_{j=1}^{L} r_j s_j)/(N_{ri}), 1)] \), which implies that \( y_i/(r_0 + \sum_{j=1}^{L} r_j s_j)/r_i = f^{-1}(s_i) \), and hence \( y_i = r_i f^{-1}(s_i)/(r_0 + \sum_{j=1}^{L} r_j s_j), \) \( i \in \mathcal{L} \). Let \( \tilde{s} = (s_1, \ldots, s_L) \), \( \tilde{y} = (y_1, \ldots, y_L) \), and \( \tilde{r} = (r_0, r_1, \ldots, r_L) \). Hence, we can express inventory as a function of service as follows:

**Definition 2.** The service-inventory mapping is defined as \( \bar{\Gamma}(\tilde{s}; \tilde{r}) = (\Gamma_1(\tilde{s}; \tilde{r}), \ldots, \Gamma_L(\tilde{s}; \tilde{r})) \), where \( \Gamma_i(\tilde{s}; \tilde{r}) = r_i f^{-1}(s_i)/(r_0 + \sum_{j=1}^{L} r_j s_j) \).

**Theorem 1.** The service-inventory mapping \( \bar{\Gamma} : \tilde{s} \in (0, 1)^L \rightarrow \tilde{y} \in (0, +\infty)^L \) is one to one and onto.

By Theorem 1, the service-inventory mapping \( \bar{\Gamma}^{-1} \) is well defined and implies an equivalence between choosing inventory levels or committing to service rates. We define \( \Gamma_i^{-1} \) as the \( i \)-th component of \( \bar{\Gamma}^{-1} \). The following proposition shows some properties of \( \bar{\Gamma} \) and \( \bar{\Gamma}^{-1} \).

**Proposition 3.** (1) \( \Gamma_j \) is increasing in \( s_j \), but decreasing in \( s_i \), where \( i \neq j \); (2) \( \Gamma_j \) is increasing in \( r_j \), but decreasing in \( r_i \), where \( i \neq j \); (3) \( \Gamma_j^{-1} \) is increasing in \( y_i \), where \( i = 1, \ldots, L \); (4) \( \Gamma_j^{-1} \) is decreasing in \( r_j \), but increasing in \( r_i \), where \( i \neq j \).

By Proposition 3, committing to a higher service rate requires an increase in inventory. An increase of the choice probability \( r_i \) of product \( i \) (e.g., by a price reduction) boosts its demand, which in turn requires an increase in inventory to preserve the precommitted service rate. In contrast, an increase of the choice probabilities or service rates of the other products reduces demand for product \( i \), and hence the inventory required to maintain the precommitted service rate. The service rate of product \( i \) increases with the inventory level of product \( i \) and also in the inventory levels of other products. This is because increasing inventories of other products reduces their stockout probabilities, which reduces demand substitution for product \( i \) and hence increases its service rate. An increase of the choice probability of product \( i \) or a decrease of the choice probabilities of other products stimulates demand for product \( i \) and hence reduces its service rate for a fixed inventory level.

We approximate the expected effective demand for product \( i \) by \( E[I_i, s_i] = E[N](r_i s_i/(r_0 + \sum_{j=1}^{L} r_j s_j)) \) and the profit \( \pi_i = E[N](r_i s_i/(r_0 + \sum_{j=1}^{L} r_j s_j)) - c_i y_i \), where \( y_i = \Gamma_i(\tilde{s}) \) for every \( i \in \mathcal{L} \). We define the approximate expected market share of product \( i \) as \( m_{si} = r_i s_i/(r_0 + \sum_{j=1}^{L} r_j s_j) \).

Note that we simply replace the product availability process \( \chi(t) \) by a constant service rate \( s_i \) in the expected effective demand for product \( i \). The constant service rate \( s_i \) roughly measures the proportion of time in \([0, T]\) that demand for product \( i \) can be met, i.e., \((1/T) \int_0^T \chi_i(t) \, dt\). To be specific, we consider a simple scenario, in which customer arrivals are evenly distributed in \([0, T]\). Then, the stockout time of product \( i \) (denoted by \( T_i \)) can be approximated by \( \min(y_i/I_i, T) \). Hence, the expected proportion of time that demand for product \( i \) can be met \((1/T) \int_0^T \chi_i(t) \, dt = E[T_i]/T \) can be approximated by \( s_i = E[\min(y_i/I_i, 1)] \).

Finally, note that \( s_i = 1 \) implies that \( y_i = +\infty \) and all demand for product \( i \) can be satisfied. Moreover, if \( s_i = 1 \) for all \( i \in \mathcal{L} \), then the effective demand for product \( i \) becomes \( E[N](r_i/(r_0 + \sum_{j=1}^{L} r_j)) \), that is, a standard attraction model.

**Proposition 4.** (1) \( m_{si} \) is increasing in \( s_j \), but decreasing in \( s_i \), where \( i \neq j \); (2) \( m_{sj} \) is increasing in \( r_j \), but decreasing in \( r_i \), where \( i \neq j \); (3) \( m_{sj} \) is increasing in \( y_j \), but decreasing in \( y_i \), where \( i \neq j \).

By Proposition 4, the market share of product \( i \) increases in its inventory level, due to a reduction in the stockout rate, but decreases in the inventory levels of other products because of reduced demand substitution for product \( i \). The market share of product \( i \) also increases in its choice probability \( (r_i) \), as we would expect.

Propositions 3 and 4 show that solving the static network generates an approximation with appropriate analytical properties. The logical behavior of this approximation suggests that it should not exhibit extreme errors. Example 1 confirms the analytical properties of Propositions 3 and 4 and shows that the static approximation also performs well numerically.

**Example 1.** We consider three products in a market with \( \tilde{r} = (1, 1, 1, 1) \). We set the inventory levels of products 2 and 3 to be \( y_2 = 250 \) and \( y_3 = 300 \) and vary the inventory level of product 1 from 150 to 250. We assume that the number of customer arrivals \( N \) is normal with mean 1,000 and standard deviation \( \sqrt{1,000} \), which approximates Poisson arrivals with mean 1,000. This approximation is used in van Ryzin and Mahajan (1999). The demand for every product is 250 on average. Hence, product 3 often satisfies its demand and has leftovers, but product 1 cannot meet its demand. The corresponding service rates \( \bar{\Gamma}^{-1}(\tilde{y}) \), which are increasing in \( y_1 \) and satisfy Proposition 3(3), are shown in Figure 4(a). The aggregate market share (i.e., \( m_{s_1} + m_{s_2} + m_{s_3} \)) is shown in Figure 4(b). We observe that the approximate market share is only slightly higher than the market share generated by simulation of the demand substitution process.

We also simulate the aggregate market share under the assumption that customer arrivals follow a Poisson process and a customer walks away if her best choice is out.
The accuracy of the approximation of the effective demand for each product determines the accuracy of the profit approximation and optimal inventory, price, and other marketing decisions. We define the percentage error of the static approximation for the effective demand of product \( i \) as

\[
\epsilon_i = \frac{E[N] \frac{r_i s_i}{r_0 + \sum_{j=1}^L r_j s_j} - E \left[ \int_0^T \frac{r_i \chi_i(t)}{r_0 + \sum_{j=1}^L r_j \chi_j(t)} dN(t) \right]}{E \left[ \int_0^T \frac{r_i \chi_i(t)}{r_0 + \sum_{j=1}^L r_j \chi_j(t)} dN(t) \right]},
\]

where \( \bar{\epsilon} = \bar{T}^{-1}(\bar{y}) \). Let \( \bar{\epsilon} = (\epsilon_1, \ldots, \epsilon_L) \).

**Example 2.** We assume that the number of customer arrivals \( N \) is normal with mean \( \lambda \) and standard deviation \( \kappa \sqrt{\lambda} \) and let \( \kappa \) vary. For the simulation, we round \( N \) up or down to get an integer value. We let \( \lambda = 1,000 \) and consider four scenarios of \((\bar{r}, \bar{y})\). In scenarios 1 and 3, inventory levels proportionally match demand for each product. In scenarios 2 and 4, inventory levels do not match demand for each product, which results in frequent stockouts and demand substitution. The percentage errors in effective demand, \( \bar{\epsilon} \), are calculated for different scenarios in Table 1.

As shown in Table 1, our static approximation is accurate when the number of customer arrivals has a low coefficient of variation (e.g., Poisson customer arrivals represented with \( \kappa = 1 \)), but the percentage errors increase in the variation of customer arrivals (\( \kappa \)). The reason for this is that replacing the product availability process \( \chi_i(t) \) by a constant service rate in the static approximation ignores the randomness of the product availability process \( \chi_i(t) \).

Furthermore, note that our approximation often overestimates effective demand. This is because, for tractability, we assume in the static approximation that a proportion of unmet customer demand goes back to the product originally requested (i.e., flows in the static network have the memoryless property). This overestimation is exaggerated as the variation of customer arrivals increases.

We check the robustness of the static approximation to changes in the number of customer arrivals in the following example.

**Example 3.** We assume Poisson customer arrivals with mean \( 1,000/\kappa \). As a fluid approximation of discrete

### Table 1. The percentage errors in Example 2.

<table>
<thead>
<tr>
<th>Scenario 1 (%)</th>
<th>Scenario 2 (%)</th>
<th>Scenario 3 (%)</th>
<th>Scenario 4 (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{r} = (1, 1, 1) )</td>
<td>( \bar{r} = (1, 1, 1) )</td>
<td>( \bar{r} = (1, 1, 2, 3) )</td>
<td>( \bar{r} = (1, 1, 2, 3) )</td>
</tr>
<tr>
<td>( \bar{y} = (250, 250, 250) )</td>
<td>( \bar{y} = (150, 300, 450) )</td>
<td>( \bar{y} = (150, 300, 450) )</td>
<td>( \bar{y} = (250, 250, 250) )</td>
</tr>
<tr>
<td>( \kappa = 0.25 )</td>
<td>(1.1, 1.1, 1.2)</td>
<td>(1.0, 0.1, 0.4)</td>
<td>(1.6, 0.0, 0.0)</td>
</tr>
<tr>
<td>( \kappa = 0.5 )</td>
<td>(1.1, 1.0, 1.0)</td>
<td>(0.9, 0.2, 0.1)</td>
<td>(1.8, 0.2, 0.2)</td>
</tr>
<tr>
<td>( \kappa = 0.75 )</td>
<td>(1.1, 1.0, 1.1)</td>
<td>(0.9, 0.2, 0.1)</td>
<td>(2.0, 0.0, 0.0)</td>
</tr>
<tr>
<td>( \kappa = 1 )</td>
<td>(1.2, 1.3, 1.3)</td>
<td>(1.2, 0.7, 0.5)</td>
<td>(2.3, 0.1, 0.1)</td>
</tr>
<tr>
<td>( \kappa = 2 )</td>
<td>(1.8, 1.8, 1.8)</td>
<td>(1.6, 1.9, 1.3)</td>
<td>(3.6, 0.5, 0.5)</td>
</tr>
<tr>
<td>( \kappa = 4 )</td>
<td>(3.5, 3.5, 3.5)</td>
<td>(3.2, 2.8, 2.7)</td>
<td>(6.9, 1.7, 1.4)</td>
</tr>
<tr>
<td>( \kappa = 6 )</td>
<td>(5.5, 5.5, 5.5)</td>
<td>(4.9, 4.6, 4.6)</td>
<td>(9, 5.4, 4)</td>
</tr>
<tr>
<td>( \kappa = 8 )</td>
<td>(7.8, 7.8, 7.8)</td>
<td>(7.6, 4.3, 2.5)</td>
<td>(9.3, 11.1, 9)</td>
</tr>
</tbody>
</table>
arrivals, we let the number of customer arrivals $N$ be normal with mean 1,000/κ and standard deviation $\sqrt{1,000/\kappa}$ (the coefficient of variation is $\sqrt{\kappa}/1,000$). Let $\kappa = 1, 2, 5, 10, 20$ (correspondingly, $\mu = 1,000, 500, 200, 100, 50$). We consider similar scenarios to those in Example 2, except that inventory levels change proportionally with the mean arrival rate. The percentage errors $\bar{e}$ for different scenarios are calculated in Table 2.

These results show that the static approximation is accurate for the case of Poisson arrivals when the mean arrival rate is large, but that accuracy declines as the mean arrival rate decreases. This may be due in part to the fluid approximation, which ignores the integrality of customer arrivals. We believe, however, that it is primarily a result of the fact that Poisson arrivals with low mean have a relatively high coefficient of variation.

In summary, our static approximation dramatically simplifies the dynamic model of demand substitution used by Mahajan and van Ryzin (2001a, b), but achieves consistent analytical properties and is reasonably accurate in predicting the mean of effective demand. Accuracy is particularly good for the case of Poisson arrivals with a high mean, which is a fairly typical case. For example, high-volume Poisson demand is a reasonable representation of customer traffic at a major retail store during Christmas. The tractability of this approximation enables us to embed demand substitution into models of price, service, and product assortment competition and predict market outcomes, as we show in §§3 and 4.

### 3. Price and Service Competition

In this section, we study a symmetric decentralized system, in which every retailer offers a single product and selects the price and service rate that maximize her profit. After establishing the uniqueness of a pure-strategy equilibrium in the price and service competition, we predict equilibrium inventory levels by the service–inventory mapping. Lippman and McCardle (1997), Mahajan and van Ryzin (2001b), and Netessine and Rudi (2003) proved the existence of a pure-strategy equilibrium in an inventory competition with demand substitution under fixed prices. A pure-strategy equilibrium in an inventory and price competition may not exist, depending on the demand-rationing rule adopted (see, e.g., Kreps and Scheinkman 1983 and Davidson and Deneckere 1986). When a mixed-strategy equilibrium has to be employed, it is hard to define a simple equilibrium inventory measure.

Our price and service competition with the service–inventory mapping gives us an indirect prediction of equilibrium inventory levels in a competitive market, in which firms compete along both operational (product availability) and marketing (price) dimensions. However, note that this prediction is premised on the assumption that all retailers use the static approximation. Moreover, note that a retailer must be able to anticipate the equilibrium service rates of the others so that she can adjust her inventory level to implement her equilibrium service rate. Hence, our results are suggestive rather than an exact representation of what would happen in the original price and inventory game with demand substitution.

To measure the competition effect, we also study a centralized system, in which a single central retailer offers all products. For simplicity, we assume that customer choices follow a logit model (i.e., $r_i = e^{-\gamma_i}$, $i \in \mathcal{X}$, and $r_0 = 1$) and unit product costs are equal (i.e., $c_i = c$).

#### 3.1. The Centralized Case

In the centralized system, a single central retailer maximizes system profit,

$$
\Pi_c = \sum_{i=1}^{L} \pi_i = E[N] \sum_{i=1}^{L} \frac{P_i r_i s_i}{1 + \sum_{j=1}^{L} r_j s_j} - c \sum_{i=1}^{L} y_i,
$$

where $y_i = r_i f^{-1}(s_i)/(1 + \sum_{j=1}^{L} r_j s_j)$. By Theorem 1, it is equivalent to choose service rates or select inventory levels. We let $\overline{p}_i = p_i - cf^{-1}(s_i)/(E[N]s_i)$ represent the price markup and $\overline{q}_i = \ln(s_i) - cf^{-1}(s_i)/(E[N]s_i)$ denote the service-quality markup of product $i$. Then,

$$
\Pi_c = E[N] \frac{\sum_{i=1}^{L} \overline{p}_i e^{\overline{q}_i}}{1 + \sum_{i=1}^{L} e^{\overline{q}_i - \overline{p}_i}}.
$$

Let

$$
\overline{q} = \max_{s \in (0, 1)} \left\{ \ln(s) - \frac{cf^{-1}(s)}{E[N]s} \right\}
$$

and

$$
s^* \in \arg \max_{s \in (0, 1)} \left\{ \ln(s) - \frac{cf^{-1}(s)}{E[N]s} \right\}.
$$

To guarantee the uniqueness of $s^*$, we first introduce a family of distributions.

### Table 2. The percentage errors in Example 3.

<table>
<thead>
<tr>
<th>Scenario 1 (%)</th>
<th>Scenario 2 (%)</th>
<th>Scenario 3 (%)</th>
<th>Scenario 4 (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{r} = (1, 1, 1, 1)$</td>
<td>$\bar{r} = (1, 1, 1, 1)$</td>
<td>$\bar{r} = (1, 1, 2, 3)$</td>
<td>$\bar{r} = (1, 1, 2, 3)$</td>
</tr>
<tr>
<td>$\bar{y} = (1/\kappa)(250, 250, 250)$</td>
<td>$\bar{y} = (1/\kappa)(150, 300, 450)$</td>
<td>$\bar{y} = (1/\kappa)(150, 300, 450)$</td>
<td>$\bar{y} = (1/\kappa)(250, 250, 250)$</td>
</tr>
<tr>
<td>$\kappa = 1$</td>
<td>$(1.3, 1.3, 1.3)$</td>
<td>$(-0.3, 0.6, -0.2)$</td>
<td>$(0.8, 0.2, -0.1)$</td>
</tr>
<tr>
<td>$\kappa = 2$</td>
<td>$(1.8, 1.8, 1.7)$</td>
<td>$(-0.3, 1.2, -0.5)$</td>
<td>$(1.7, 0.6, 0.2)$</td>
</tr>
<tr>
<td>$\kappa = 5$</td>
<td>$(3.1, 2.9, 3)$</td>
<td>$(1.1, 2.7, -1.5)$</td>
<td>$(3.2, 1.4, 0.6)$</td>
</tr>
<tr>
<td>$\kappa = 10$</td>
<td>$(4.4, 4.3, 4.1)$</td>
<td>$(1.5, 4.1, -2.5)$</td>
<td>$(5, 2.7, 1.5)$</td>
</tr>
<tr>
<td>$\kappa = 20$</td>
<td>$(4.5, 4.3, 4.5)$</td>
<td>$(-3.2, 7, -1.9)$</td>
<td>$(4.4, 5, 2.3)$</td>
</tr>
</tbody>
</table>
DEFINITION 3. A positive random variable \( N \) has a scaled increasing likelihood ratio (SILR) iff \( \theta_2 N \) is larger than \( \theta_1 N \) in likelihood ratio order (denoted as \( \theta_1 N \gtrsim \theta_2 N \) for \( \theta_2 > \theta_1 > 0 \).

It is easy to check that the SILR family includes Fisher’s \( F \), gamma, inverse gamma, lognormal, and Weibull distributions, but not the normal. Moreover, because normal distributions have an increasing hazard rate (IFR), but some gamma and Weibull distributions do not belong to the IFR family (Müller and Stoyan 2002), neither the SILR nor IFR are subsets of each other. However, we note that the IFR family is contained in the IGFR family (Lariviere and Porteus 2001) and prove in the following proposition that the SILR family is also a proper subset of IGFR.

PROPOSITION 5. (1) \( \text{SILR} = \{ N \mid \text{the density of } \log(N) \text{ is log-concave} \}; \) (2) \( \text{SILR} \subset \text{IGFR} \).

We can now use the SILR property to show that the service level that maximizes the service-quality markup is unique.

PROPOSITION 6. (1) There exists \( s^* \) in \( (0, 1) \), which maximizes \( \ln(s) - c f^{-1}(s)|(\mathbb{E}[N]|s) \); (2) \( s^* \) is unique if \( N \) has the SILR property; (3) \( \bar{q}^* < 0 \); (4) \( s^* \) is decreasing in \( c \); (5) \( s^* \) is unchanged when \( N \) is scaled by a constant.

Proposition 6 shows that SILR demand is a sufficient condition for the uniqueness of \( s^* \). Furthermore, in all of our numerical experiments using the normal distribution, we also observed a unique \( s^* \). Hence, for the remainder of the paper we assume that \( s^* \) is unique. Under this condition, we can characterize the optimal centralized solution as follows.

THEOREM 2. In the centralized system: (1) the optimal price markups are positive and equal; (2) the optimal service rate is \( s^* \) for all products.

The equal price markup result of Theorem 2 follows from the fact that the logit model implies equal cross-price elasticities (i.e., changing the price of one product has an equal impact on demand for all other products). This property often holds for horizontally differentiated products. For example, Swatch offers hundreds of watches by making use of cosmetic modular design (e.g., changing colors and shapes of faces and wristbands) and prices them uniformly (see Hopp and Xu 2005 for details). Other horizontally differentiated products with uniform pricing include soft drinks, cosmetic products, and music CDs. The property of equal price markups was also observed by Anderson and de Palma (1992) for a nested-logit model, Aydin and Ryan (2000) for a logit model, and Hopp and Xu (2005) for a Bayesian logit model. By Claim 2 of Theorem 2, the central planner should set the service rate of every product at level \( s^* \), which is independent of prices and is decreasing in product cost.

We define \( a_i = L e^{\bar{p}_i} \) as the market power of the central retailer, which enables us to express the system profit as \( \Pi_c = \mathbb{E}[N](a_i \bar{p} e^{-\bar{p}})/(1 + a_i e^{-\bar{p}}) \).

THEOREM 3. (1) The optimal price markup \( \bar{p}_c^* \) is uniquely determined by \( \bar{p}_c^* = 1 + a_c e^{-\bar{p}_c} \); (2) The optimal profit for the central retailer \( \Pi_c^* = \mathbb{E}[N|(\bar{p}_c^* - 1) \) and the optimal profit for every product is given by \( \pi_i^* = \mathbb{E}[N|L(\bar{p}_i - 1)] \); (3) \( \bar{p}_c^* \) and \( \Pi_c^* \) are increasing in \( L \) and \( a_c \), but decreasing in \( c \); (4) \( \pi_i^* \) is increasing in \( a_i \), but decreasing in \( L \) and \( c \).

Because the central retailer’s market power is increasing (decreasing) in the number of products (product cost), the optimal profit markup and aggregate profit are increasing (decreasing) in the number of products (product cost), but the profit per product is decreasing with the saturation of the market. The optimal market share is \( ms^*_c = (1/L)(a_c e^{-\bar{p}_c} / (1 + a_c e^{-\bar{p}_c})) \), the inventory level per product is \( y_c^* = (1/L)(a_c e^{-\bar{p}_c} / (1 + a_c e^{-\bar{p}_c}))(f^{-1}(s^*)/s^*) \), the aggregate optimal market share is \( MS^*_c = a_c e^{-\bar{p}_c} / (1 + a_c e^{-\bar{p}_c}) \), and the aggregate optimal inventory level is \( Y_c^* = (a_c e^{-\bar{p}_c} / (1 + a_c e^{-\bar{p}_c}))(f^{-1}(s^*)/s^*) \). Note that whereas the optimal service rate is independent of price, the optimal inventory levels do depend on price.

PROPOSITION 7. (1) \( MS^*_c \) and \( Y_c^* \) are increasing in \( L \), but decreasing in \( c \); (2) \( ms^*_c \) and \( y_c^* \) are decreasing in \( L \) and \( c \).

Proposition 7 shows that the aggregate optimal market share and inventory level increase as the number of products increases, but decrease in product cost. In contrast, the optimal market share and inventory per product also decrease in the number of products, due to market saturation.

3.2. The Decentralized Case

We now consider the case where every retailer offers a single product and seeks to maximize her individual profit. Recall that \( \pi_i = \mathbb{E}[N](p_i e^{-p_i} s_i)/(1 + \sum_{j=1}^L e^{-p_j} s_j) - c y_i = \mathbb{E}[N](\bar{p}_i e^{\bar{p}_i})/(1 + \sum_{j=1}^L e^{\bar{p}_j} s_j), i \in \mathcal{I}. \) To model competition, we study a game in which every retailer chooses a price and commits to a service rate. We focus on a simultaneous single-stage game (i.e., price and service rate are determined simultaneously). A two-stage game, with a service-rate competition as the first stage and price competition as the second stage, can be studied similarly (see, e.g., Bernstein and Federgruen 2004b). Let \( a_d = e^{\bar{p}_d} \) be a retailer’s market power. Note that \( a_i = L a_d. \)

THEOREM 4. (1) There exists a unique equilibrium, that is, every retailer chooses service rate \( s^* \) and price markup \( \bar{p}_d^* \), which satisfies \( 1/\bar{p} = 1 - a_d e^{-\bar{p}} / (1 + L a_d e^{-\bar{p}}) \); (2) The equilibrium aggregate profit is \( \Pi_c^* = \mathbb{E}[N|L(\bar{p}_c - 1) \) and the equilibrium profit per product is \( \pi_i^* = \mathbb{E}[N|L(\bar{p}_i - 1), i \in \mathcal{I} ; (3) \bar{p}_c^* \) and \( \pi_i^* \) are increasing in \( a_d \), but decreasing in \( c \) and \( L \); (4) \( \Pi_c^* \) is increasing and converges to \( \Pi_c^* \) as \( L \to +\infty \); (5) \( \bar{p}_c^* < \bar{p}_c^* \).

We note that the equilibrium service rate is independent of price decisions, which is a consequence of the logit demand assumption. For other types of demand (e.g., linear), this property does not hold (see Bernstein and
By Theorem 4, competition lowers both the price and the profit of every retailer, but the aggregate profit increases as more retailers enter the market. The logit demand model implies that a product is more attractive to a segment of the market than the other products. Hence, retailers do not need to completely rely on lowering price to attract customers, which allows a positive aggregate profit even when there are infinite retailers in a market.

Note that although it is equivalent for a central retailer to set inventory levels or commit to service rates (due to the service-inventory mapping introduced in §2), this equivalence does not hold in a competitive market because service rates are jointly determined by competitors’ inventory levels via demand substitution. In fact, it is well known that there may not exist a pure-strategy equilibrium in an inventory and price competition with demand substitution/rationing (i.e., the Bertrand-Edgeworth paradox, see, e.g., Deneckere and Peck 1995 for additional discussion). Because of this, it is difficult to define a simple equilibrium inventory measure with a mixed-strategy equilibrium. By focusing on a price and service competition, we are able to obtain a tractable pure-strategy equilibrium, such that we can easily recover required inventory and predict market outcomes. Note that the problem of nonexistence of a pure-strategy equilibrium in an inventory and price game disappears if unmet demand is backlogged—that is, when there is no demand substitution. Under the backlog assumption, inventory decisions of different retailers do not affect each other’s demand, and hence a pure-strategy equilibrium can be established (see, e.g., Bernstein and Federgruen 2004a and Kirman and Sobel 1974).

In the decentralized case, the equilibrium market share per product is \( ms^*_a = \frac{a_d e^{-\tilde{p}_d}}{1 + L_a e^{-\tilde{p}_d}} \), the equilibrium inventory level per product is \( y^*_a = \frac{(a_d e^{-\tilde{p}_d})(1 + L_a e^{-\tilde{p}_d})}{(f^{-1}(s^*)/s^*)} \), the aggregate equilibrium market share is \( MS^*_a = L_a e^{-\tilde{p}_d}/(1 + L_a e^{-\tilde{p}_d}) \), and the aggregate equilibrium inventory level is \( Y^*_a = (L_a e^{-\tilde{p}_d}/(1 + L_a e^{-\tilde{p}_d}))(f^{-1}(s^*)/s^*) \). We can further characterize the properties of the equilibrium market share and inventory level as follows.

**Proposition 8.** (1) \( MS^*_a \) and \( Y^*_a \) are increasing in \( L \), but decreasing in \( c \); (2) \( Y^*_a \) converges to \( f^{-1}(s^*)/s^* \) as \( L \to +\infty \); (3) \( ms^*_a \) and \( y^*_a \) are decreasing in \( L \) and \( c \); (4) \( MS^*_a > MS^*_c \) and \( mc^*_a > ms^*_a \); (5) \( Y^*_a > Y^*_c \) and \( y^*_a > y^*_c \).

Proposition 8 implies that competition results in more inventory than in the centralized case, which is consistent with the results of Lippman and McCardle (1997), Mahajan and van Ryzin (2001b), and Netessine and Rudi (2003). However, because price is exogenous in these earlier models, overstock is driven by inventory competition. In contrast, the driver of more inventory in our model is price competition, which occurs because competition lowers price, which in turn increases demand, and hence, needed inventory. If price markups are fixed in our game, then the centralized and decentralized systems achieve the same outcome because committing to the service rate \( s^* \) is always the best choice for every retailer. This result provides a possible explanation for why retailers often focus on service and adopt cost-based pricing policies (e.g., setting price as cost plus a fixed margin), namely, that it lessens competitive pressure on price and inventory. However, because it depends on the logit demand model, this conclusion must be interpreted with caution.

Proposition 8 also shows that although inventory level, market share, and profit of individual retailers decrease in the number of retailers, the aggregate inventory level, market share, and profit *increase* in the number of retailers. Theorem 8 of Lippman and McCardle (1997) and Theorem 4 of Mahajan and van Ryzin (2001b) showed that as the number of retailers goes to infinity, the aggregate inventory level converges to a constant, at which aggregate profit is zero. The aggregate inventory level in our model also converges to a constant, but unlike in previous models, our aggregate profit converges to a positive constant. This is because committing to the service rate \( s^* \) is always the best choice for every retailer, and hence the negative effect of competition is only from price decisions. However, because the price competition is not very intense for the logit demand model, a positive aggregate profit remains.

The sensitivity of markup \( \tilde{p}^* \), retailer profit \( \pi^* \), aggregate profit \( \Pi^* \), retailer market share \( ms^* \), aggregate market share \( MS^* \), retailer inventory \( y^* \), and aggregate inventory \( Y^* \) with respect to the number of retailers \( L \) are summarized in Table 3.

<table>
<thead>
<tr>
<th></th>
<th>( \tilde{p}^* )</th>
<th>( \pi^* )</th>
<th>( \Pi^* )</th>
<th>( ms^* )</th>
<th>( MS^* )</th>
<th>( y^* )</th>
<th>( Y^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Centralized system</td>
<td>↑</td>
<td>↓</td>
<td>↓</td>
<td>↑</td>
<td>↓</td>
<td>↓</td>
<td>↑</td>
</tr>
<tr>
<td>Decentralized system</td>
<td>↓</td>
<td>↓</td>
<td>↑</td>
<td>↓</td>
<td>↑</td>
<td>↓</td>
<td>↑</td>
</tr>
</tbody>
</table>

### 4. Product Assortment Competition

In this section, we consider two producers, each with a product line, who compete on product assortment, service rates, and prices. Product line extension is an important issue in category management. On one hand, extending a product line increases the possibility that customers find a fit, which results in higher price premium and revenue. On the other, product proliferation increases operational costs (inventory carrying cost and other overhead costs) and stockout chances, which result in customer dissatisfaction and lost sales. As in the previous application, we assume that demand follows a logit model. The adoption of a logit model yields a monotonic ranking of products, in terms of quality markups, for the product assortment problem. Aydin and Ryan (2000), Hopp and Xu (2005), and van Ryzin and Mahajan (1999) found similar ranking indices for their product assortment models with logit demand. All
these models are monopolistic and do not consider product assortment competition. We first consider a monopoly model in §4.1 and then study a duopoly model in §4.2.

4.1. The Monopoly Case

We denote the menu of potential products that could be introduced by a producer by \( \{ (q_i, c_i) \}_{i=1}^{\tilde{n}} \), where \( \tilde{n} \) is the maximum number of products that the producer is able to offer, \( q_i \) is the quality index, and \( c_i \) is the unit cost of product \( i \). Hence, the customer choice probabilities are given by \( r_i = e^{q_i - p} \). Letting \( |\mathcal{I}| \) be the number of elements in subset \( \mathcal{I} \), we represent the overhead cost of offering product line \( \mathcal{I} \) by \( h_i g_i(|\mathcal{I}|) \), where \( \mathcal{I} \) is the set of products offered, \( g_i \) is assumed to be an increasing function of the product line length, and \( h_i \) is a constant.

The monopolist seeks to maximize her profit,

\[
\Pi = \sum_{i \in \mathcal{I}} \pi_i = \sum_{i \in \mathcal{I}} \left[ E[N] \left( \frac{p_i r_i s_i}{1 + \sum_{j \in \mathcal{I}} p_j s_j} - c_i N_i \right) - h_i g_i(|\mathcal{I}|) \right],
\]

where \( y_i = r_i f^{-1}(s_i)/(1 + \sum_{j \in \mathcal{I}} r_j s_j) \), \( i \in \mathcal{I} \). We define \( \bar{p}_i = p_i - c_i f^{-1}(s_i)/(E[N] s_i) \) (\( i \in \mathcal{I} \)) as the price markup and \( \bar{q}_i = q_i + \ln(s_i) - c_i f^{-1}(s_i)/(E[N] s_i) \) (\( i \in \mathcal{I} \)) as the quality markup for product \( i \). With these, we can express the profit function as \( \Pi = E[N] \sum_{i \in \mathcal{I}} \bar{p}_i e^{\bar{q}_i} \)/\( (1 + \sum_{j \in \mathcal{I}} e^{\bar{q}_j}) - h_i g_i(|\mathcal{I}|) \). Let \( s^* \in \arg \max_{s \in (0,1)} \left[ \ln(s) - c_i f^{-1}(s)/(E[N] s) \right] \) represent the optimal service level and \( \bar{q}^*_i = q_i + \ln(s^*_i) - c_i f^{-1}(s^*_i)/(E[N] s^*_i) \) represent the optimal quality markup for product \( i \). Define \( \{ \bar{q}^*_i \}_{i=1}^{\tilde{n}} \) to be a decreasing rearrangement of \( \{ \bar{q}^*_i \}_{i=1}^{\tilde{n}} \) and \( \mathcal{L} = \{(1,2,\ldots,n)\} \) to be the set of products with the \( n \) largest optimal quality markers. If \( \{ \bar{q}^*_i \}_{i=1}^{\tilde{n}} \) strictly decreases, then \( \mathcal{L}^* \) is unique.

**Theorem 5.** (1) The optimal price markups are positive and equal for all products; (2) For product \( i \in \mathcal{I} \), the optimal service rate is \( s^*_i \); (3) \( \mathcal{L}^* \) is an optimal product line for some \( n \).

Claim 3 of Theorem 5 shows that a monopolist should offer the products with the largest optimal quality markups. Similar assortment results were obtained by Proposition 2 in Hopp and Xu (2005) and Theorem 1 of van Ryzin and Mahajan (1999). Hopp and Xu (2005) assumed unlimited capacity and gave a monotonic ranking of products, measured by quality minus unit product cost. Van Ryzin and Mahajan (1999) considered capacity constraints, but assumed no demand substitution in consumer behavior and equal product costs. Their monotonic ranking of products is captured by only quality indices. Our model extends this insight to the situation where demand uncertainty, as well as quality and cost factors, affect the monotonic ranking for the product assortment problem.

**Example 4.** We consider two products with quality and cost indices \( (q_1, c_1) \), where \( q_1 < q_2 \) and \( c_1 < c_2 \). We assume that the monopolist offers only one product and \( q_2 > q_1 > c_2 - c_1 \). If we ignore demand uncertainty and assume unlimited capacity, by Proposition 2 of Hopp and Xu (2005), the monopolist should offer product 2 because \( q_1 - c_1 < q_2 - c_2 \).

Now, we assume that customer demand follows a normal distribution with mean \( \mu \) and standard deviation \( \sigma \). Let

\[
\Delta(\sigma) = \max_{i \in (0,1)} \left\{ \ln(s) - c_i f^{-1}(s)/(E[N] s) \right\} - \max_{i \in (0,1)} \left\{ \ln(s) - c_i f^{-1}(s)/(E[N] s) \right\}.
\]

The monopolist should offer product 1 if \( q_2 - q_1 < \Delta(\sigma) \) and product 2 if \( q_2 - q_1 > \Delta(\sigma) \). Figure 5 shows how the product assortment decision changes with \( \sigma \) when \( c_1 = 2, c_2 = 4, \) and \( \mu = 1,000 \).

As shown in Figure 5(a), the optimal service rates decrease as demand uncertainty increases. Product 2 is offered when demand uncertainty is low because a high service rate can be achieved without stocking too much. When demand uncertainty increases, product 1 becomes more attractive because it costs less in stock to hedge increased demand uncertainty than does product 2. As demand uncertainty keeps increasing further, it becomes expensive to maintain high service rates for both products, product 1 loses its relative cost advantage, and product 2 resumes its role as the better choice.

Note that although demand uncertainty affects product assortment decisions, the magnitude of its effect is very small. This may help explain the numerical findings of Maddah and Bish (2007), which assumes a logit demand model but uses a different approximation. They found that offering the most popular products may not be optimal, but still gives a profit very close to the optimal profit (see Tables 1–3 therein). Our example suggests that calibrating the popularity index of products with demand uncertainty may solve this puzzle.

We can now write the monopoly profit function as \( \Pi = E[N] (\tilde{p} S_n(n) e^{-\tilde{\sigma}})/(1 + S_n(n) e^{-\tilde{\sigma}}) - h_i g_i(n), \) where \( S_n(n) = \sum_{i=1}^{n} e^{x_i} \) is called the product line power, the market power generated by the product line, and \( n \) is the product line length. Theorem 5 simplifies the product assortment problem to one of determining the product line length. Similar to Theorem 3, the optimal price markup \( \tilde{p}^*_n(n) \) satisfies \( \tilde{p}^*_n(n) = 1 + S_n(n) e^{-\tilde{\sigma}} \) given product line length \( n \). To determine the optimal product line length \( n^*_n \), we only need to maximize \( \Pi = E[N] (\tilde{p}^*_n(n) - 1) - h_i g_i(n), \) which is a special case of the problem studied in Hopp and Xu (2005).

To derive conditions under which the optimal product line length is unique, we define an efficient overhead cost \( \tilde{g}_n(a_n) = g_i(S_n^{-1}(a_n)), \) where \( a_n \in R^+ \). For example, if \( S_n(n) = n \) and \( g_i(n) = n^{1/c_i} \), then \( \tilde{g}_n(a_n) = a_1^{1/c_i} \). Furthermore, if \( k < 1 \), then \( \tilde{g}_n \) is convex. Note that both the product line power \( S_n \) and the overhead cost \( g_i \) determine the shape of \( \tilde{g}_n \).

Finally, we let \( M_{n}^* = S_n(n^*_n) e^{-\tilde{\sigma}}/(1 + S_n(n^*_n) e^{-\tilde{\sigma}}) \) represent the optimal aggregate market share and \( Y_{n}^* = (e^{-\tilde{\sigma}}/(1 + S_n(n^*_n) e^{-\tilde{\sigma}})) \sum_{i \in \mathcal{I}} e^{x_i} (f^{-1}(s_i)/s_i^*) \) represent the optimal aggregate inventory level.
Figure 5. (a) Optimal service rates $s_1^*$ and $s_2^*$; (b) Product assortment decision.

Proposition 9. (1) If $g_c$ is linear or convex and $c_s \in \mathbb{R}^+$, the optimal product line length $n^*_c$ and the optimal price markup $\tilde{p}_l^c$ are unique; (2) The optimal price markup $\tilde{p}_l^c$ and the optimal market share $MS_l^c$ are increasing in the optimal product line length $n^*_c$; (3) If product line $X^*$ is majorized by product line $\tilde{X}^*$ and the overhead function is unchanged, then product line $\tilde{X}^*$ is more profitable; (4) If $s_l^* = s^*$ for any $i$, then $Y^*_l$ is increasing in the optimal product line length $n^*_c$; (5) $n^*_c$ is decreasing in $h_c$.

Claims 1 and 2 of Proposition 9 are consistent with Observation 1 of Mahajan and van Ryzin (2001a).

4.2. The Duopoly Case

We now turn to the case where two producers compete on the basis of price, service, and product assortment. We index the producers by $l \in \{1, 2\}$ and let $-l$ represent the competitor of producer $l$. We make use of the notation introduced in §4.1 with each parameter subindexed by $l$ or $-l$. For example, producer $l$'s menu is expressed as $(d_{l,i}(c_{l,w}), s_{l,i})$ and producer $l$'s profit is given by $\pi_l = E[N](\sum_{i \in J_l} \tilde{p}_{l,i} e^{t_i - \tilde{p}_{l,i}})/(1 + \sum_{i \in J_l} e^{t_i - \tilde{p}_{l,i}} - h_l g_l([J_l]))$.

We represent the competition as a sequential game. In the first stage, each producer decides her product line. In the second stage, they engage in a price and service competition after observing each other’s product line. Because the price and service subgame is similar to the game in §3.2, we briefly summarize the results and focus on the product assortment subgame.

In the price and service subgame, we fix producer $l$’s product line $J_l$ and denote producer $l$’s market power as $a_l = \sum_{i \in J_l} e^{t_i}$. With this, we can characterize the equilibrium as follows.

Theorem 6. (1) There exists a unique equilibrium in the price and service subgame, in which producer $l$ chooses for each product $i \in J_l$ a service rate $s^*_i$, such that $s^*_i = \arg \max_{s \in (0, 1]} \ln(s) - c_{l,i} f^{-1}(s)/(E[N] s)$, and a price markup $\tilde{p}_l^i(a_l; a_{-l})$, which satisfies $\tilde{p}_l^i - 1 - a_l e^{-\tilde{p}_l^i}/(1 + a_{-l} e^{-\tilde{p}_l^i}) = 0$, $l = 1, 2$; (2) The equilibrium profit for producer $l$ is $\pi_l^*(a_l; a_{-l}) = E[N](\tilde{p}_l^i(a_l; a_{-l}) - 1) - h_l g_l([J_l])$; (3) $\pi_l^1(a_1; a_{-1})$ and $\tilde{p}_l^1(a_l; a_{-l})$ are increasing in $a_l$, but decreasing in $a_{-l}$.

Let $ms_l^i(a_l; a_{-l}) = a_l e^{-\tilde{p}_l^i}/(1 + \sum_{i \in J_l} a_l e^{-\tilde{p}_l^i})$ denote the equilibrium market share and $y_l^i(a_l; a_{-l}) = (e^{-\tilde{p}_l^i}/(1 + \sum_{i \in J_l} a_l e^{-\tilde{p}_l^i})) \sum_{i \in J_l} e^{t_i}(f^{-1}(s_{l,i}^*))/s_{l,i}$ represent the inventory level of producer $l$.

Proposition 10. (1) The equilibrium market share $ms_l^i$ is increasing in $a_l$, but decreasing in $a_{-l}$; (2) If $s_{l,i}^* = s_i^*$, then $y_{l,i}^*$ is increasing in $a_l$, but decreasing in $a_{-l}$.

By Proposition 10, a producer’s market share and inventory level increase in her market power, which is determined by her product line. However, extending a product line also increases overhead cost, which presents a trade-off similar to those observed in quality competitions (see, e.g., Banker et al. 1998). We study this trade-off by means of the assortment subgame, in which each producer chooses her product line prior to engaging in the price and service competition.

Theorem 7. (1) It is a dominating strategy for producer $l$ to offer $X_{l,n_i}$ for some $n_i$; (2) There exists a pure-strategy Nash equilibrium for the assortment subgame.

Theorem 7 implies that an equilibrium product line is composed of the products with the largest optimal quality markups. Notice, however, that because the overhead cost $g_l$ may be concave due to economies of scope in product assortment, the supermodularity theory used in the proof of Theorem 7 cannot be replaced by a quasi-concavity argument, such as that used by Anderson and de Palma (1992) to prove the existence of an equilibrium in a product variety game.

Denote the least equilibrium of the assortment subgame by $(n_{l,1}^*, n_{l,2}^*)$, such that $n_{l,2}^* \leq n_{l,1}^*$ and $n_{l,1}^* \leq n_{l,2}^*$ for any equilibrium $(n_{l,1}^*, n_{l,2}^*)$. Define the corresponding equilibrium price markups as $(\tilde{p}_{l,1}^*, \tilde{p}_{l,2}^*)$, the market shares as $(ms_{l,1}^*, ms_{l,2}^*)$, and the profit functions by $(\pi_{l,1}^*, \pi_{l,2}^*)$. The notation for the
The efficient overhead cost measures the overhead cost and the efficient overhead cost for each producer. To derive conditions under which a unique equilibrium is guaranteed, we similarly define the product line power $S_i$ and the efficient overhead cost $\tilde{g}_i(a_i)$ for each producer $i$. The efficient overhead cost measures the overhead cost required to achieve a market power index.

**Theorem 8.** If $\tilde{g}_i$ is linear or convex and $n_i \in R^+$, there exists a unique pure-strategy Nash equilibrium of the assortment subgame.

The following example shows that in general, the assumption of convexity of $\tilde{g}_i$ cannot be dropped.

**Example 5.** Consider a symmetric game with $E[N] = 1,000$, $h_1 = 100$, $S_i(n_i) = n_i$, and $g_i(n_i) = \sqrt{n_i}$. Hence, $\tilde{g}_i(a_i) = \sqrt{a_i}$. If $n_i \in R^+$, the least equilibrium is $n_i = 5.05$, $n_i = 193.02$, with $\tilde{p}_1 = 1.27$, $\tilde{p}_2 = 3.69$, $ms_1 = 21.47\%$, $ms_2 = 72.94\%$, $\pi_1 = 49$, and $\pi_2 = 1,310$. The greatest equilibrium can be derived by exchanging the subindices. Clearly, a concave overhead cost function $\tilde{g}_i$ permits multiple equilibria.

Concavity of the overhead cost function implies economies of scope in product line length, which could be the result of logistics efficiencies in the distribution channel, the producer’s negotiation power, and/or the product design technology (see, e.g., Hopp and Xu 2005 for a discussion of economies of scope generated by modularity in product design). Convexity of the overhead cost function implies diseconomies of scope, which could be the result of allowance costs and shelf-space constraints.

To enable us to describe the properties of the product assortment equilibrium, we rewrite

$$ms_i'(n_i; n_{-i}) = \frac{S_i(n_i)e^{-\tilde{p}_i}}{1 + \sum_{i'=1}^{2} S_i(n_i)e^{-\tilde{p}_{i'}}}$$

and

$$y_i'(n_i; n_{-i}) = \frac{e^{-\tilde{p}_i}}{1 + \sum_{i'=1}^{2} S_i(n_i)e^{-\tilde{p}_{i'}}} \sum_{i' \in F_i} \tilde{e}_{i',i} f^{-1}(s_{i',i}).$$

**Proposition 11.** (1) The equilibrium price markup $\tilde{p}_i$ and the equilibrium market share $ms_i'$ are increasing in $n_i$, but decreasing in $n_{-i}$; (2) If $s_{i',i} > s_{i',i'}$ for any $i$, then $y_{i'}$ is increasing in $n_i$, but decreasing in $n_{-i}$; (3) $n_i$ is increasing in $h_{-i}$, but decreasing in $h_i$.

Proposition 11 implies that product line extension increases a producer’s price markup, market share, and required inventory level, but decreases price markup, market share, and inventory level of the competitor.

By Claim 5 of Proposition 8, the aggregate inventory level in a decentralized system is always higher than the aggregate inventory level in a centralized system when the number of products is taken exogenously. We establish a similar result for a symmetric assortment game, in which $q_{1,i} = q_{2,i} = q_i$, $c_{1,i} = c_{2,i} = c_i$, $h_1 = h_2 = h_i = h$, and $g_1 = g_2 = g_0 = g$. Hence, $s_{1,i} = s_{2,i} = s_i = s^*$ and $\tilde{q}_{1,i} = \tilde{q}_{2,i} = \tilde{q}_i$. Without loss of generality, we assume that $\tilde{q}^* = 0$, which implies $S_i(n) = S_i(n) = S_i(n) = n$. Recall that the optimal aggregate market share is $MS_i^* = n^*e^{-\tilde{p}_i} / (1 + n^*e^{-\tilde{p}_i})$ and the optimal aggregate inventory level is $Y_i^* = (n^*e^{-\tilde{p}_i} / (1 + n^*e^{-\tilde{p}_i}))(f^{-1}(s^*) / s^*)$ in the monopoly case. In the duopoly case, the optimal aggregate market share of both producers is $MS_i^* = (n^*e^{-\tilde{p}_i} + n_2^*e^{-\tilde{p}_2}) / (1 + n^*e^{-\tilde{p}_i} + n_2^*e^{-\tilde{p}_2})$ and the optimal aggregate inventory level is $Y_i^* = ((n^*e^{-\tilde{p}_i} + n_2^*e^{-\tilde{p}_2}) / (1 + n^*e^{-\tilde{p}_i} + n_2^*e^{-\tilde{p}_2}))(f^{-1}(s^*) / s^*)$.

**Proposition 12.** Assume that $g$ is linear or convex, $n_i \in R^+$, and $n_i \in R^+$. (1) For the symmetric assortment game, there exists a unique symmetric pure-strategy Nash equilibrium (i.e., $n_i = n_2 = n_3^*$), which results in a symmetric price equilibrium $\tilde{p}_1 = \tilde{p}_2 = \tilde{p}_3$; (2) $n_3^* < n_2^*$; (3) $\tilde{p}_d^* > \tilde{p}_s^*$; (4) $ms_3 = ms_3^* < MS_3^*$; and $y_3^* = y_3^* > Y_3^*$; (5) $MS_d^* < MS_3^*$ and $Y_d^* < Y_3^*$; (6) If $n_k \leq 2n_3$, then $MS_k^* > MS_3^*$ and $Y_k^* > Y_3^*$.

Competition in the decentralized system limits the ability of a producer to increase price to compensate for the overhead cost of a larger product assortment, and hence discourages product line extension. As a result, both producers should reduce product line length and play the “puppy dog” strategy in the product assortment competition to soften the damage of price competition because price is strategically complement (Tirole 1990). Hence, the product line length for each producer is shorter than if they were a monopolist, which results in less required inventory for each producer, as suggested by Claims 2 and 4 of Proposition 12. Because adopting mass customization often facilitates product line extension, by a similar argument as above, price competition deters the adoption of mass customization (see, e.g., Alptekinoğlu and Corbett 2008 and Mendelson and Parkaktürk 2008).

Claim 6 of Proposition 12 shows that the aggregate inventory level in a competitive market is higher than in a monopolistic market if the total number of products in the competitive market is larger. Example 6 suggests that this assumption is very likely.

**Example 6.** We assume a linear overhead cost function $g(n) = hn$, $h \in [0.03, 0.3]$. Figure 6(a) plots product line lengths $n_1$ and $n_2$ and suggests that the condition of Claim 6 of Proposition 12 holds. This implies that the aggregate market share in a competitive market is larger than in a monopolistic market, as shown in Figure 6(c). Finally, Figure 6 shows that $\tilde{p}_s^* > \tilde{p}_d^*$ (Claim 3 of Proposition 12).

Note that the condition of Claim 6 of Proposition 12 may not hold for small values of $h$. For example, when $h = 0.0003$, $n_2 = 116.3$ and $n_2^* = 256.5 > 2n_2 = 232.6$. However, even under this case, $MS_2^* = 91\% > MS_2^* = 77\%$, and hence Claim 6 still holds. This is because competition
limits the pricing power of a producer even if she offers a long product line. Hence, the price markup in a competitive market is much lower than the monopolistic price $\bar{p}_d^* = 1.94 < \bar{p}_c^* = 4.34$, which results in a higher aggregate market share in the competitive market. Finally, it is possible that Claim 6 breaks down when $n^* > 2n^*_d$ under a concave overhead function (a counterexample is given in the online appendix).

5. Conclusions and Future Research

In this paper, we develop a static approximation of dynamic customer demand substitution behavior by representing customer decisions with a fluid network model, in which nodes represent products and flows represent customer demands. With this network, we define a service-inventory mapping and approximate effective demand and market share for each product in an equilibrium sense. Because of its static form, our approximation is vastly simpler than previous dynamic representations (e.g., Mahajan and van Ryzin 2001a, b), and therefore permits inclusion of demand substitution behavior in models of product pricing, stocking, and variety.

To illustrate the usefulness of a static demand substitution model, we apply our approximation to two applications, both of which assume that customer preferences are represented by a logit model. First, we study a simultaneous price and service competition between single-product retailers. We establish the existence of a unique pure-strategy Nash equilibrium, where the equilibrium service rate is determined by demand uncertainty and unit product cost and is independent of prices and other parameters. The service equilibrium determines the equilibrium inventory levels by means of a service-inventory mapping. We find that competition results in lower prices and higher inventory levels in a decentralized system than in a centrally planned system that offers the same products. However, aggregate profit and inventory level increase to positive constants as the number of retailers goes to infinity.

Second, we study a duopolistic competition in price, service, and product assortment. We model this as an assortment subgame, followed by a price and service competition. In the assortment subgame, the duopolists consider customer perception, unit product cost, and demand uncertainty, and decide whether each product should be offered. We rank products according to their optimal quality markups and show that both producers should offer the assortment of products with the largest optimal quality markups. We establish existence of a pure-strategy Nash equilibrium for the product assortment competition and show that convexity of the overhead cost function is sufficient for uniqueness. We find that endogeneity of the product assortment can lead to a shorter product line for each producer than if they were a monopolist, due to the fact that cutting product lines may avoid price competition, but the total number of products in a competitive market is likely larger than in a monopolistic market, which results in a higher aggregate inventory level.

The above applications are two examples where a simple demand substitution model is useful. Other research areas where our approximation may enhance modeling and analysis are shelf-space management in retail environments and component substitution in assemble-to-order systems.
Moreover, although we have shown that our static model yields a reasonably accurate quantitative approximation of the means of the effective demands when the coefficient of variation of the number of total customer arrivals is low (e.g., Poisson arrivals with high mean), our numerical experiments are still preliminary. Further analysis of the accuracy of our approximation is needed under other applications and/or under different criteria. For example, an interesting test would be to see how accurate our approximation is for the entire distribution of the effective demand. This can be measured by Kolmogorov distance.

In the context of specific applications, it is also of interest to evaluate the accuracy with which the approximation predicts variables that depend on demand. For example, in the price and service competition, we would like to know how well our approach approximates equilibrium inventory levels. Unfortunately, analyses like this are likely to be extraordinarily difficult because it is precisely the intractability of the original problem that motivated the need for the approximation in the first place. Short of a very tedious numerical embedding of the algorithm of Mahajan and van Ryzin (2001a, b) into a game with price competition, we do not see any way to do this.

Beyond incorporating customer demand substitution into various operations problems and evaluating the accuracy of our approximation, there are two major areas that deserve research attention. First, we note that our work, like that of Lippman and McCardle (1997), Mahajan and van Ryzin (2001b), and Netessine and Rudi (2003), predicts higher inventory levels in a competitive market. Moreover, our work, as well as that of Maddah and Bish (2007), Mahajan and van Ryzin (2001a), and van Ryzin and Mahajan (1999), characterizes the optimal product assortment. However, none of these studies made any empirical tests of their predicted market outcomes. Anupindi et al. (1998) and Kök and Fisher (2007) did use data to test their conclusions about how to estimate demand censored by stockouts and how to set product assortments, but neither paper offered a direct empirical test of the competition effects on inventory and product assortment outcomes. Because, as we have seen in this paper, competition may dramatically change price, inventory, and product assortment decisions, more empirical studies are needed to validate these results and evaluate their magnitude.

Second, we point out that our model, as well as most of the others in the literature on product variety, assume that customers are rational and justify demand in simple economic terms. However, evidence from the marketing literature suggests that customers often behave irrationally and are influenced by the shopping environment and other subjective factors (Fitzsimons 2000). To develop useful support tools for product decisions, more detailed study of the nuances surrounding customer demand substitution behavior is needed.

6. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/.

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