Capital Asset Valuation and Depreciation for Stochastically Deteriorating Equipment

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Abstract
This paper develops a capital asset valuation theory for stochastically deteriorating equipment in the absence of an active secondary (resale) market to provide prices that reflect economic value. With this theory, we define an economic depreciation schedule that satisfies two properties. First, book value of equipment depreciated according to this schedule equals conditional expected economic value; and second, the schedule is fair in that expected capital gain (or loss) is zero.

Introduction
Previous literature has developed a theory of capital asset valuation under two major assumptions: first, that equipment deteriorates deterministically, and second, that there is an active secondary (or resale) market for the equipment. This paper relaxes these two major assumptions to develop a theory of economic valuation for stochastically deteriorating equipment for which there is no active secondary market. Additionally, we use this theory of economic valuation to provide a theory of equipment depreciation. This work extends the methods and results of an earlier paper [9], where we relaxed the second assumption but kept the first. Our work is based on the user-cost theory of machine replacement which posits that users of durable goods derive value from the services provided by durables rather than from the durables themselves. User-cost theory is well documented in the economic literature and provides the basis for virtually all machine replacement models. Jorgenson [12] examined macroeconomic data and concluded that user-cost theory provides an explanation for empirically observed demand for durables. Previous work aimed at providing a theory of value for durables has also been based on user-cost theory. Wykoff [19], for example, showed that one implication of user-cost theory was a relationship between used equipment prices and economic rent. He also performed an econometric study of automobile markets and concluded that the prices of new cars are explained by this relationship, but the prices of used cars are not. Subsequently, Johnson [8] pointed out some flaws in Wykoff's original study and concluded that existing data for used car prices are indeed explained by Wykoff's relationship. This work has also been applied to the problem of defining economic depreciation. See [7] and [13]. Hulten and Wykoff [7], for example, modeled the economic depreciation of cohorts when individual assets fall off (retire) at different times.
The case of stochastic deterioration is important because it accurately models many real systems (see [1, chap. 1] for a survey). The case of equipment for which there is no competitive secondary market is important because, for many important capital assets, it is realistic. Brealey and Myers [2] (page 11), for example, observed that market prices measure an asset’s economic value when there is a competitive market for the asset, but that the market for many corporate assets is too thin for the competitive condition to hold. Specific examples of corporate assets for which no secondary market exists include packaging machinery and telecommunications equipment [10].

This paper makes two main contributions. First, we derive a theory of value for stochastically deteriorating equipment with no secondary market and show that this generalizes Wykoff’s earlier result [19]. Second, we use this theory of value to derive a companion theory of conditional expected economic depreciation. We argue that, for fairness, a depreciation schedule should satisfy the condition that expected capital gains equal zero and show that our theory provides fair depreciation schedules.

In section 2 of the paper, we present the Stochastic Machine Replacement Problem (SMRP). We develop a theory of capital asset valuation in section 3, and apply it to a special case of the SMRP, the classic “used car” example of Howard [6], in section 4. Finally, in section 5, we use this theory of economic value to develop a theory of economic depreciation for the SMRP.

**Stochastic Machine Replacement Problem (SMRP)**

Suppose that each of the machines owned by a risk neutral firm occupies one of \( n+1 \) observable states, ordered from "best" or "new" (state 0) to "worst" (state \( n \)), where \( n \) is some prescribed positive integer. The demand for the services of each machine is assumed to be constant over an infinite horizon. In any year, a machine is either kept or replaced. A machine in state \( n \) provides no capacity and will always be replaced. When a machine is replaced, it can be replaced by a machine in any state (except in the worst state \( n \)). If a machine is kept, it will either remain in the same state or deteriorate to an unknown but less desirable state the following year. The state transitions due to the operation of equipment occur according to a Markovian transition matrix with diagonal entries strictly less than one. In other words, the probability of a machine remaining in the same state the following year when kept is less than one. The transition probabilities are denoted by:

\[
\pi(i,j) = \text{the probability that a machine in state } i \\
\text{that is kept in any year will be in state } j \\
\text{the following year (} i, j = 0, 1, \ldots, n \text{), where} \\
\pi(i,j) = 0 \text{ if } j < i, \text{ and } \pi(i,i) < 1 \text{ for all } i < n.
\]

Costs and receipts associated with each machine are stationary, i.e., dependent on the state of the machine but not on the year in which they occur. Maintenance, operating, and obsolescence costs, as well as purchase costs, occur at the beginning of the year; salvage receipts occur at the end of the year. Since we are not assuming the existence of an active or efficient market for used equipment, the prices for used machines are given exogenously. The economic parameters are denoted by:

\[
p_i = \text{the purchase price of a machine in state } i \\
(i = 0, 1, \ldots, n-1); \\
s_i = \text{the salvage value of a machine in state } i \\
(i = 0, 1, \ldots, n); \\
m_i = \text{the opportunity cost (including maintenance, operating, and obsolescence) of}
\]
keeping a machine in state \( i \) \((i = 0,1,\ldots,n-1)\);
\[
\delta = \frac{1}{1+r}, \text{ where } r \text{ is the effective annual interest rate (} \delta \text{ is called the discount factor).}
\]

We make two assumptions about these economic parameters:

1. \( 0 < \delta < 1 \);

2. \( p_i \geq s_j \geq 0 \ (i = 0,1,\ldots,n-1; \ j = i,\ldots,n) \).

Assumption 1 implies that \( 0 < r < \infty \) since \( \delta = 1/(1+r) \). Assumption 2 implies that a machine can never be salvaged for more than the market price of a machine in the same or a more desirable state.

The management objective for this risk neutral firm is to minimize the discounted expected sum of disbursements less receipts over the infinite horizon. This problem can be formulated and solved in a standard way using dynamic programming. See [14, p. 36] for a representative example. In the next section, we present a solution approach that yields explicit values of assets over time.

Before continuing, we address an obvious limitation of our model: it has large information requirements, i.e., exogenously provided transition probabilities, purchase prices, and salvage values for equipment in the various possible states. For the purposes of this paper, we assume knowledge of these parameters to provide as general a model as possible. In practice, however, making further assumptions about these parameters, for instance by using the MAPI maintenance cost models of Terbohr [15-18] and the "one-step-down" probability models of [1], would greatly reduce the information requirements of the model. In this manner, our approach could be made into the basis for a practical method for determining capital asset values and setting depreciation schedules. The specifics of such a method are outside the scope of this paper.

**Capital Asset Valuation For The SMRP**

Suppose a machine in state \( \tau \) has just been purchased, where \( \tau \in \{0,1,\ldots,n-1\} \). We need to determine an optimal replacement policy for this machine that specifies a replacement decision (of either keep or replace) for each state that the machine can transit to, namely states \( \tau \) through \( n \). Since a machine in state \( \tau \) has just been purchased, rather than replaced, it must be optimal to keep a machine in state \( \tau \). Also, all machines that reach state \( n \) must be replaced. Corresponding to states \( \tau+1 \) through \( n-1 \), there are at most \( 2^n-\tau \) possible replacement options to follow for machines that reach these states since there are two replacement options per state. Let \( RP_\tau \) denote the set of these replacement options:

\[
R P_\tau = \{ \sigma_\tau \text{ is a vector of size } n - \tau - 1 \text{ such that } \sigma_\tau(i) = 0 \text{ or } 1, \ i = \tau + 1,\ldots,n-1 \},
\]

where a keep decision for a machine in state \( i \) is represented by \( \sigma_\tau(i) = 1 \), and a replace decision for a machine in state \( i \) is represented by \( \sigma_\tau(i) = 0 \). By definition, \( \sigma_\tau(\tau) = 1 \). As of the time the machine is in state \( i \in \{ \tau,\ldots,n-1 \} \), the expected present cost of owning a machine in state \( i \) and operating it until it is replaced according to replacement policy \( \sigma_\tau \in R P_\tau \) is denoted by \( E P C_\tau(\sigma_\tau) \) and is specified by the following recursion:

\[
E P C_\tau(\sigma_\tau) = \sigma_\tau(i) \left[ m_i + \delta \sum_{j=i}^{n} \pi(i,j)E P C_\tau(\sigma_\tau) \right] + [1-\sigma_\tau(i)](-s_i),
\]

where \( E P C_n(\sigma_\tau) = -s_n \). By rearranging, \( E P C_\tau(\sigma_\tau) \) may also be expressed as:
\[
\begin{align*}
EPC_i(\sigma_t) &= \frac{\sigma(i)[m_i + \delta \sum_{j=i+1}^{n} \pi(i,j)EPC_j(\sigma_j)] + (1 - \sigma(i))(-s_i)}{1 - \delta \sigma(i)\pi(i,i)} \\
\alpha_i(\sigma_t) &= \frac{n-1}{\sigma(i)[1 + \delta \sum_{j=i+1}^{n} \pi(i,j)\alpha_j(\sigma_j)]}
\end{align*}
\]

As of the time the machine is in state \( \tau \), the expected present cost of purchasing a machine in state \( \tau \) and operating it until it is replaced according to replacement policy \( \sigma \in RP \) is therefore \( p + EPC(\sigma) \).

The expected annual equivalent cost of purchasing a machine in state \( i \) and operating it until it is replaced according to replacement policy \( \sigma \in RP \) is the value of the constant annual cost, denoted by \( y^*(\sigma_t) \), distributed over the expected replacement cycle that results from \( \sigma_p \) and yields the same expected present cost as \( p + EPC(\sigma) \). In other words, \( y^*(\sigma_t) \) is the amortized annual cost of the machine over the expected replacement cycle that results from \( \sigma_t \).

To specify \( y^*(\sigma_t) \), suppose a dollar is to be charged for every year a machine, currently in state \( i \), or one of its future successors, is to be operated over the expected replacement cycle that results from \( \sigma_t \). Let \( \alpha_i(\sigma_t) \) denote the expected present cost of this cash flow stream, as of the current time that the machine is in state \( i \). It follows that:

\[
\alpha_i(\sigma_t) = \sigma(i)[1 + \delta \sum_{j=i}^{n} \pi(i,j)\alpha_j(\sigma_j)],
\]

where

\[
\sigma(n-1) = \sigma(n-1)[1 + \delta \sigma(n-1)\pi(n-1,n-1)].
\]

By rearranging, this quantity may also be expressed as:

\[
\alpha(i,\sigma_t) = \frac{n-1}{\sigma(i)[1 + \delta \sum_{j=i+1}^{n} \pi(i,j)\alpha_j(\sigma_j)]}
\]

where

\[
\alpha_n(\sigma_t) = \sigma(n-1)[1 + \delta \sigma(n-1)\pi(n-1,n-1)].
\]

By definition of \( y^*(\sigma_t) \),

\[
y^*(\sigma_t) = \rho + EPC(\sigma_t).
\]

Therefore:

\[
y^*(\sigma_t) = \frac{p + EPC(\sigma_t)}{\alpha_i(\sigma_t)}.
\]

The SMRP is a special case of a general sequential decision process for which the optimal replacement policy is known to be invariant or time-independent [3]. This implies that the SMRP will have an optimal replacement policy that is cyclic. Given that machines are always replaced with a machine in state \( \tau \) if they are replaced according to replacement policy \( \sigma_p \) it then follows that the optimal replacement policy for a machine purchased in state \( \tau \) is found by solving:

\[
y^*(\sigma_t) = \min_{\sigma_t \in RP} y^*(\sigma_t).
\]

It is then straightforward to determine the optimal state \( \tau^* \) that the replacement machines should occupy when purchased by solving:

\[
\tau^* = \arg \min_{0 \leq \tau \leq n-1} y^*(\tau),
\]

where \( \arg \min \) refers to the argument \( \tau \) that
achieves the minimum $y^*$. The optimal expected annual equivalent cost is given by:

$$y^* = y^*_{t^*}. \quad (6)$$

The derivation of $y^*$ is needed to specify the economic value of a capital asset subject to stochastic deterioration as assumed in the SMRP. Define $y^*(i), 0 \leq i \leq n$, by:

$$y^*(i) = \begin{cases} 
\max & y^* - m_i + \delta \sum_{j=i+1}^{n} \pi(i,j)y^*(j) \\
1 - \delta \pi(i,i) \\ s_i, i = 0, 1, \ldots, n-1
\end{cases} \quad (7)$$

where $y^*(n) = s_n$. It can be shown using algebraic arguments, not repeated here, that $y^*(t^*) = p_{t^*}$. Note that the computation of $y^*(i)$ in closed form is summarized by expressions (1) - (7).

**Theorem 1:** The economic value of owning a machine in state $i$, $0 \leq i \leq n$, that is subject to stochastic deterioration as assumed in the SMRP, is $y^*(i)$, as given by expression (7).

**Proof:** Since $y^*$ is the expected annual equivalent cost of operating a machine over its expected replacement cycle, $y^*$ is the expected marginal cost of owning a machine in an arbitrary state $i$. For a firm that maximizes expected profits, expected marginal revenue is equal to expected marginal cost. Therefore, $y^*$ is the expected value of the services provided by a machine in state $i$, and $y^* - m_i$ is the expected economic rent of the services a machine provides for a year.

By definition, the economic value of a machine in an arbitrary state $i$ is the larger of the economic value of the machine if it is replaced and the machine's economic value if it is kept. The value of a machine in state $i$ that is kept must equal the expected value of the economic rent of the services it provides for a year, $y^* - m_i$, plus the expected discounted economic value of owning the machine one year later. Applying a recursive argument in reverse from states with known economic values, $y^*(i)$ is shown to measure the economic value of a machine in state $i$.

Theorem 1 generalizes the results of Wykoff [19], who assumed that equipment deteriorates deterministically and that secondary (resale) markets are perfectly competitive. Wykoff showed that the secondary market price of a machine is equal to the machine's economic rent plus the discounted secondary market price of a machine one year older.

**Corollary 1:** The theory of secondary market valuation for deterministically deteriorating equipment of Wykoff [19] is a special case of Theorem 1.

**Proof:** For the special case of deterministic deterioration, Theorem 1 implies that $y^*(i) = y^* - m_i + \delta y^*(i+1)$, if an $i$ year old machine is to be kept, where $y^* - m_i$ is the economic rent of an $i$ year old machine as in Theorem 1. If a competitive secondary market exists, it follows by a simple arbitrage argument that $y^*(i)$ is the secondary market price of an $i$ year old machine.

Note that the prices $y^*(i)$ can also be computed by solving a finite linear program. This follows since optimal invariant replacement policies in stochastic dynamic programming can be computed using linear programming [3]. Because the closed form expression of (7) is
easier to express and more efficient to compute, we omit the linear program formulation. In addition, we need the form of expression (7) for the economic argument of Theorem 1.

The "One-Step-Down" Deterioration Model and Howard's "Used Car" Example

Variations on the "one-step-down" deterioration model have been used extensively to represent probabilistic deterioration and damage [1]. In the basic model, the system either stays in the same state, declines by one state, or goes to the last state (catastrophic failure). This model is a special case of the SMRP with the additional restrictions on the transition probabilities that \( \pi(i,j) = 0 \) if \( j = i+2, \ldots, n-1 \).

An example of the "one-step-down" version of the SMRP is the well known "used car" example of Howard [6]. Howard considered the problem of replacing used automobiles that are subject to stochastic collapse over a time interval of ten years. Every three months, the current situation is reviewed and a decision is made whether to keep the present car or to trade it in at that time for another "used car." To keep the number of states finite, a 10 year old car is considered to be worn out and is automatically replaced. When a machine is kept, there is some probability, dependent only on the age of the machine, that it will collapse, i.e., be regarded as equivalent to a 10 year old car. Otherwise, if the car is kept and it does not collapse, it ages one quarter. Purchase costs, salvage values, and opportunity costs depend only on the age of the car. Howard modeled this example using dynamic programming, and considered both the undiscounted and discounted versions. His goal was to optimally determine the ages at which a car should be replaced, as well as the age of the replacement vehicle.

Since transitions in Howard’s example are limited to the next state or the terminal state when a car is kept, it follows that \( \pi(i,i) = 0 \) for all states \( i \), and that \( \pi(i,n) = 1 - \pi(i,i+1) \) for \( i < n-1 \), in the SMRP version of this example. We now use the theory developed in the preceding section to determine prices that measure economic value for a “used car” of Howard’s example.

The following numbers were derived from the data given in [6, p. 56, 89], and shown in Table 1. Modifications to the original data of Howard were necessary because he assumed that decisions were reviewed every quarter, while we find it convenient to assume that decisions are reviewed every year.

We can determine the prices \( y^{**} \) for this SMRP by using expressions (1) - (7). These prices are:

\[
\begin{align*}
y^{**(0)} &= 2000.00, \\
y^{**(1)} &= 1050.00, \\
y^{**(2)} &= 796.84, \\
y^{**(3)} &= 599.26, \\
y^{**(4)} &= 440.00, \\
y^{**(5)} &= 315.89, \\
y^{**(6)} &= 216.79, \\
y^{**(7)} &= 160.00, \\
y^{**(8)} &= 135.00, \\
y^{**(9)} &= 110.00, \\
y^{**(10)} &= 80.00, \\
y^{*} &= 602.16.
\end{align*}
\]

These values are also shown in Table 1. Corresponding to the prices \( y^{**(i)} \) is a stationary optimal replacement policy deduced from the structure of the prices. This policy is to always replace cars 7 years or older with a 4 year old car. Note that our annual analysis differs somewhat from Howard’s quarterly analysis, which resulted in an optimal policy to buy a 3 year old car and replace it when it is 6 and 3/4 years old. Any differences in the results simply reflect the differing assumptions about review frequency.

In the next section, we address the issue of how to "fairly" depreciate equipment that is subject to stochastic deterioration, like Howard’s "used cars." To motivate this, we refer to [9], where we devised an economic theory of depreciation for the deterministic analogue of the
Table 1 Automobile Replacement Data

<table>
<thead>
<tr>
<th>Age in Years $i$</th>
<th>Purchase Price ($P_i$)</th>
<th>Salvage Value ($S_i$)</th>
<th>Opportunity Cost ($m_i$)</th>
<th>Survival Probability $\pi(i,i+1)$</th>
<th>Economic Value $y^{**}(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2,000</td>
<td>1,600</td>
<td>208</td>
<td>0.994</td>
<td>2,000.00</td>
</tr>
<tr>
<td>1</td>
<td>1,300</td>
<td>1,050</td>
<td>254</td>
<td>0.969</td>
<td>1,050.00</td>
</tr>
<tr>
<td>2</td>
<td>900</td>
<td>710</td>
<td>304</td>
<td>0.925</td>
<td>796.84</td>
</tr>
<tr>
<td>3</td>
<td>600</td>
<td>480</td>
<td>349</td>
<td>0.858</td>
<td>599.26</td>
</tr>
<tr>
<td>4</td>
<td>440</td>
<td>330</td>
<td>399</td>
<td>0.789</td>
<td>440.00</td>
</tr>
<tr>
<td>5</td>
<td>360</td>
<td>255</td>
<td>445</td>
<td>0.719</td>
<td>315.89</td>
</tr>
<tr>
<td>6</td>
<td>300</td>
<td>200</td>
<td>500</td>
<td>0.610</td>
<td>216.79</td>
</tr>
<tr>
<td>7</td>
<td>250</td>
<td>160</td>
<td>564</td>
<td>0.418</td>
<td>160.00</td>
</tr>
<tr>
<td>8</td>
<td>210</td>
<td>135</td>
<td>653</td>
<td>0.145</td>
<td>135.00</td>
</tr>
<tr>
<td>9</td>
<td>170</td>
<td>110</td>
<td>810</td>
<td>-</td>
<td>110.00</td>
</tr>
<tr>
<td>10</td>
<td>-</td>
<td>80</td>
<td>-</td>
<td>-</td>
<td>80.00</td>
</tr>
</tbody>
</table>

Annual Discount Factor, $\delta = 0.89$, Terminal Age of the Car, $n = 10$.

SMRP by setting the yearly decline in book value equal to the decline in economic value as measured by the deterministic analogue of the prices $y^{**}(i)$. There the decline in state (or age) one year later for a machine that is kept is completely deterministic; the machine simply ages one year. Thus, the definition of an economic depreciation schedule for this deterministic model was straightforward once the prices $y^{**}(i)$ were determined.

This leads to the issue of whether and how to use the prices $y^{**}(i)$ to define an economic depreciation schedule for Howard's used car that is subject to stochastic rather than deterministic deterioration. This definition is complicated by the fact that depreciation is a time-based concept while the prices $y^{**}(i)$ are state-based, and time and state do not correspond directly in the stochastic environment as they do in the deterministic environment. Thus, although the states at which a machine is replaced in an observable process are known with complete certainty (even in advance), the chronological age of the machine at replacement is not known in advance.

**Economic Depreciation for the SMRP**

Suppose the government wished to impose an economic depreciation schedule for a machine subject to stochastic deterioration as in the SMRP. Because of stochasticity, the government cannot determine a priori the precise value of such a machine at every future point in time. In principle, the government could determine an a posteriori economic depreciation schedule by inspecting equipment each year to determine its state. There are, however, at least two drawbacks to this idea. The first is obvious: large scale inspections of equipment are impractical for the government to carry out because of the huge cost involved. Second, according to the tax code, tax policy in the U.S. requires that tax payers know a priori the tax consequences of their decisions to acquire assets.

Given that the government must define an a priori economic depreciation schedule for...
stochastically deteriorating equipment, what qualities should such a schedule have? To help answer this question, we again refer to the more straightforward deterministic model of [9], where it is observed that economic schedules have two obvious characteristics. First, they measure decline in economic value as measured by \( y^{**} \). Second, they are fair in the sense that a piece of equipment that is depreciated over its full economic lifetime according to such schedules has a book value equal to its market salvage value. In other words, neither a capital gain nor loss is realized using the deterministic economic depreciation schedule over the economic lifetime of the asset.

For a machine subject to stochastic deterioration, however, a capital gain or loss can result at the time of replacement for an a priori economic depreciation schedule since it is not known (in advance) with certainty when a machine will be replaced. It may be possible, though, for an a priori schedule to have the property that if the firm buys and depreciates this type of machine many times or if many firms buy identical machines of this type, then on average book value equals market value, i.e., the expected capital gain (or loss) is zero.

One simple way of defining an a priori economic schedule for the SMRP is for the government to determine a priori the expected price of a machine in a particular year by using \( y^{**} \) and the transition probabilities. The expected economic depreciation schedule could then be defined by taking differences of these expected prices. Such a depreciation schedule is not fair according to our definition, however, because the expected capital loss at replacement due to this schedule is not necessarily zero (e.g., this quantity is positive when computed using the data of Howard's example). This raises the following question. Can we define an a priori economic depreciation schedule for stochastically deteriorating equipment that measures decline in expected economic value and also is fair in that it results in an expected capital gain (or loss) of zero? The answer is yes.

To see this, note that another way to define an a priori economic depreciation charge of a machine in any year is by setting the decline in book value equal to the expected decline in economic value of the machine in that year, given that the machine has not been previously replaced. Suppose that a machine in state \( \tau \) has just been purchased in year 0, where \( \tau \in \{0, \ldots, n-1\} \). Let \( K \) be defined as the set of states for which it is optimal to keep a machine (\( \tau \in K \)). Define \( \pi(i; \tau) \) to be the probability that a machine purchased in state \( \tau \) in year 0 is not replaced prior to year \( t \) and is in state \( i \) in year \( t \). These probabilities can be defined recursively by:

\[
\pi(i; \tau) = \begin{cases} 
1, & \text{if } i = \tau, \\
0, & \text{if } i = \tau + 1, \ldots, n, 
\end{cases} \quad (8)
\]

\[
\pi(i; \tau) = \sum_{j \in K} \pi(j; \tau) \pi(j, i), \quad t = 1, 2, \ldots \quad (9)
\]

In addition, define \( \pi(i; \tau) \) to be the conditional probability that a machine is in state \( i \) in year \( t \), given that it was purchased in state \( \tau \) in year 0 and has not been replaced by the end of year \( t \). These conditional probabilities are specified recursively with the following expressions:

\[
\pi(i; \tau) = \begin{cases} 
1, & \text{if } i = \tau, \\
0, & \text{if } i = \tau + 1, \ldots, n, 
\end{cases} \quad (10)
\]

\[
\pi(i; \tau) = \frac{\pi(i; \tau)}{\sum_{k \in K} \pi(k; t)}, \quad i \in K, \quad t = 1, 2, \ldots \quad (11)
\]
The conditional expected economic depreciation, \( d(t; \tau) \), to be charged in year \( t \) for a machine purchased in state \( \tau \) in year 0, is defined as:

\[
d(t; \tau) = \sum_{i \in K} \sum_{j=1}^{n} \pi_{i,j}(i|\tau)[y^{**}(i) - \sum_{i} \pi(i,j)y^{**}(j)], \quad t = 1, 2, \ldots
\]

(12)

The vector \( d(t; \tau) \) is infinite dimensional; that is \( t \) has no upper bound, because there is a finite probability that a machine will last beyond any fixed horizon, due to the possibility of non-deterioration of state. Since \( \pi(i,i) < 1 \) for all \( i \in K \), however, the machine will be replaced with probability 1 as \( t \to \infty \). Note that the use of conditional probability \( \pi_{i,j}(i|\tau) \) is a reasonable way to define \( d(i; \tau) \) since \( d(t; \tau) \) will be used in year \( t \) only if the machine has not been replaced prior to year \( t \).

To illustrate this definition of economic depreciation, we again return to the data of Howard's example. Following the optimal replacement policy of the SMRP, suppose we buy a 4 year old car at time \( t = 0 \). We can then calculate the conditional expected economic depreciation charges for this car using expressions (8) - (12). Doing this, we obtain:

\[
d(1;4) = \pi(4,5)[y^{**}(4) - y^{**}(5)] + (1 - \pi(4,5))[y^{**}(4) - y^{**}(10)]
\]

\[
= 0.789[440 - 315.89] + (1 - 0.789)[440 - 80] = 173.88,
\]

\[
d(2;4) = \pi(5,6)[y^{**}(5) - y^{**}(6)] + (1 - \pi(5,6))[y^{**}(5) - y^{**}(10)]
\]

\[
= 0.719[315.89 - 216.79] + (1 - 0.719)[315.89 - 80] = 137.54,
\]

\[
d(3;4) = \pi(6,7)[y^{**}(6) - y^{**}(7)] + (1 - \pi(6,7))[y^{**}(6) - y^{**}(10)]
\]

\[
= 0.610[216.79 - 160] + (1 - 0.610)[216.79 - 80] = 87.99.
\]

The expected capital gain (or loss) for this schedule is then:

\[
(1 - \pi(4,5))[p_4 - d(1;4) - s_{10}] + \pi(4,5)(1 - \pi(5,6))[p_4 - d(1;4) - d(2;4) - s_{10}] + \pi(4,5)\pi(5,6)(1 - \pi(6,7))[p_4 - d(1;4) - d(2;4) - d(3;4) - s_{10}] + \pi(4,5)\pi(5,6)\pi(6,7)[p_4 - d(1;4) - d(2;4) - d(3;4) - s_7] = 0.
\]

This example suggests that the conditional expected economic depreciation schedule is fair.

To demonstrate that this is always true, denote the difference between book value (computed using the conditional expected economic depreciation schedule of expression (12)) and market salvage value when a machine is replaced, over a time horizon of \( T \) years, by \( [BV-MSV](T; \tau) \), assuming again that a machine in state \( \tau \) is purchased in year 0. If \( [BV-MSV](T; \tau) \) is less (greater) than zero, then a capital gain (loss) is realized within the \( T \) years. It follows that \( [BV-MSV](T; \tau) \) is a random variable since it is uncertain when the machine will be replaced over the period of \( T \) years, if at all. Let \( E[BV-MSV](T; \tau) \) denote the expected value of \( [BV-MSV](T; \tau) \). Since it is not guaranteed that the machine will be replaced within a given finite period of time, the machine's economic lifetime is potentially infinite. Therefore, to calculate the expected capital gain (or loss) of a machine over its economic lifetime, we compute the limit of \( E[BV-MSV](T; \tau) \) as \( T \to \infty \), which is denoted by \( ECG(\tau) \). If \( R \) denotes the set of states for which it is optimal to replace a machine (\( n \in R \)), then:
\[
E[BV-MSV](T; \tau) = \\
\sum_{t=1}^{T} \sum_{i \in R} \sum_{k \in K} (\pi_{t,i}(k; \tau) \pi(k,i)) \{ y^{**}(\tau) - \sum_{j=1}^{t} d(j; \tau) - s_i \},
\]

\[
ECG(\tau) = \\
\lim_{T \to \infty} \sum_{t=1}^{T} \sum_{i \in R} \sum_{k \in K} (\pi_{t,i}(k; \tau) \pi(k,i)) \{ y^{**}(\tau) - \sum_{j=1}^{t} d(j; \tau) - s_i \}.
\]

**Proposition 1:**

\[
ECG(\tau) = \lim_{T \to \infty} \sum_{j \in K} \pi_{\tau}(j; \tau) y^{**}(j).
\]

**Proof:** See the appendix.

**Corollary 2:** The expected capital gain (or loss) resulting from the conditional expected economic depreciation schedule of expression (12) is zero.

**Proof:** Since \( \pi(i,i) < 1 \) for all \( i \in K \) and \( K \) is finite, it must be true that:

\[
\lim_{T \to \infty} \sum_{j \in K} \pi_{\tau}(j; \tau) = 0.
\]

Assumption (2) and expression (7) imply that \( y^{**}(j) \leq p_j, j \in K \). Since \( K \) is finite, there exists a uniform upper bound on \( y^{**}(j) \) for all \( j \in K \), namely \( \max p_j (j \in K) \). Therefore, it follows from Proposition 1 that \( ECG(\tau) = 0 \).

Thus, an a priori economic depreciation schedule can be defined for stochastically deteriorating equipment that measures expected decline in economic value and is also fair in that it results in no expected capital gain (or loss). The conditional expected economic depreciation schedule of expression (12) is such a schedule.

**Further Research**

This paper represents a first step toward a theory of economic valuation and depreciation for an important class of equipment replacement problems that are subject to stochastic deterioration. A number of directions in which this research might proceed are possible. As we did with the "one-step-down" deterioration model, particular forms of stochastic deterioration can be assumed and their implications explored. For example, stochastic models involving lower Hessian matrices, where machines can degrade by at most one state per period, have been studied in [1] and [5]. In addition, models using IFR (increasing failure rate) matrices have been used in the machine maintenance and replacement literature to describe the deterioration process [4].

Other interesting problems that can be studied via a stochastic machine replacement model are problems involving risk and the lemon problem. A risk sensitive utility function could be formulated in terms of the mean and variance of the cumulative discounted reward. This would result in a quadratic objective function in the infinite-horizon linear program, which can still be solved as a linear complementarity problem [11]. In the lemon problem, the aging process is deterministic, but there is some probability that a new machine is defective and results in higher maintenance costs than a nondefective machine.

Previous research reviewed in the introduction examined capital asset valuation assuming a perfect competitive market for the assets. Our work has assumed there is no market for capital assets, the opposite market extreme. Further research is called for to examine capital asset valuation for the intermediate case of a competitive asset market that is imperfect.
Acknowledgments

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References


Appendix

Proposition 1: $ECG(\tau) = \lim_{T \to \infty} \sum_{j \in K} \pi_T(j; \tau)y^{**}(j)$.

Proof: The expected value of $[BV-MSV](T; \tau)$, $E[BV-MSV](T; \tau)$, is:

$$
  T \sum \sum \{ \sum_{t=1}^{T} \pi_{t-1}(k; \tau)\pi(k,i)\} [y^{**}(\tau) - \sum_{j=1}^{T} d(j; \tau) - s_i].
$$

It then follows that $E[BV-MSV](T; \tau)$ is equal to:

$$
  y^{**}(\tau) - \sum_{t=1}^{T} \pi_{t-1}(j; \tau)d(t; \tau) - \sum_{t=1}^{T} \sum_{j \in K} \pi_{t}(k; \tau)\pi(k,i)s_i = \sum_{t=1}^{T} \sum_{k \in K} \{ \sum_{j \in K} \pi_{t-1}(i; \tau[y^{**}(i) - \sum_{j=1}^{n} \pi(i,j)y^{**}(j)] - \sum_{t=1}^{T} \sum_{i \in K} \{ \sum_{j \in K} \pi(i,j)y^{**}(j) - \sum_{j \in K} \pi(i,j)y^{**}(j) \}.
$$
\[\sum_{t=1}^{T} \sum_{i \in R} \sum_{k \in K} \pi_{i-1}(k; \tau) \pi(k, i) s_i = \sum_{i \in K} \sum_{t=1}^{T} \pi_0(i; \tau) y^{**}(i) - \sum_{i \in K} \sum_{t=1}^{T} \pi_{i-1}(i; \tau) y^{**}(i) + \]

\[\sum_{t=1}^{T} \sum_{j \in K} \pi_{t-1}(i; \tau) y^{**}(i) + \]

\[\sum_{t=1}^{T} \sum_{i \in K} \sum_{j \in K} \pi_{i-1}(i; \tau) \pi(i, j) y^{**}(j) + \]

\[\sum_{t=1}^{T} \sum_{i \in R} \sum_{k \in K} \pi_{i-1}(k; \tau) \pi(k, i) s_i = \]

\[\sum_{t=1}^{T} \sum_{i \in K} \sum_{k \in K} \pi_{i-1}(i; \tau) y^{**}(i) + \]

\[\sum_{j \in K} \pi_{T}(j; \tau) y^{**}(j) .\]

Thus, \( ECG(\tau) = \lim_{T \to \infty} \sum_{j \in K} \pi_{T}(j; \tau) y^{**}(j) . \)

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