Generalized Imputed Salvage Values

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ABSTRACT

This paper studies the classical problem of evaluating end effects when the study period is shorter than the asset’s economic life. By relating the concept of imputed salvage value to classical micro-economic theory, we make three contributions. First, we generalize the concept of imputed salvage value, normally defined only for assets whose maintenance and operation costs are independent of age, to include assets with arbitrary operating and maintenance costs. Second, we show that generalized imputed salvage values are equivalent to the stationary dual prices of an infinite-horizon linear program, thereby providing explicit computational formulae. Third, we argue that imputed salvage values (not market salvage values) correctly measure economic value and, hence, appropriately evaluate end effects.

1. INTRODUCTION

A classical problem in engineering economics is to reconcile the fact that an asset’s economic life may be longer than the study period with the fact that economic comparisons must be based on the economic life (which may not coincide with the study period). A frequently encountered situation where this problem arises is when evaluating two capital assets whose economic lives differ. The following data present an example in which alternatives $A_1$ and $A_2$ have economic lives of five years and three years, respectively.

<table>
<thead>
<tr>
<th>End of Year</th>
<th>$A_1$ Cash Flow</th>
<th>$A_2$ Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-20,000</td>
<td>-12,000</td>
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<tr>
<td>1</td>
<td>-5,000</td>
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<td>2</td>
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<tr>
<td>3</td>
<td>-5,000</td>
<td>-7,000 + 3,000</td>
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<tr>
<td>4</td>
<td>-5,000</td>
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<td>5</td>
<td>-5,000 + 3,000</td>
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</table>
Each alternative has a salvage value of $3,000 at the end of its economic life.

Two equivalent and frequently recommended procedures deal with this problem by avoiding the need to evaluate economic end effects. First, one can compute the annual equivalent costs of each alternative over its economic life. Thus, for alternative A1, the annual equivalent costs would be computed over a 5-year period, and those for alternative A2 would be computed over a 3-year period. Once this calculation is made, the two alternatives can be compared directly on the basis of annual equivalent costs. Second, one can consider using a study period whose length is a common multiple of economic life-times. In this example, the study period would be 15 years. Since the end of the study period coincides with each alternative's replacement cycle, an evaluation of economic end effects is unnecessary.

In this paper, we consider another method, namely using a shorter study period that does not necessarily coincide with all replacement cycles. The shorter study period method, also frequently recommended, is more complicated in that it requires evaluation of economic end effects. On the other hand, it is more easily applicable to situations in which shorter study periods most accurately capture the effects of rapidly changing technology.

A procedure that engineering economy textbooks frequently recommend is to use a study period whose length equals that of the shorter economic life (3 years in this case) and to use the market salvage value of the longer-lived asset at the end of the study period to evaluate end effects. If the market salvage value of asset A\(_1\) at the end of three years is $6,000, the modified cash flows over the 3-year study period are:

<table>
<thead>
<tr>
<th>End of Year</th>
<th>A(_1) Cash Flow</th>
<th>A(_2) Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-20,000</td>
<td>-12,000</td>
</tr>
<tr>
<td>1</td>
<td>-5,000</td>
<td>-7,000</td>
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<tr>
<td>2</td>
<td>-5,000</td>
<td>-7,000</td>
</tr>
<tr>
<td>3</td>
<td>-8,000 + 6,000</td>
<td>-7,000 + 3,000</td>
</tr>
</tbody>
</table>

To evaluate the two alternatives, compute either present value or annual equivalent cost.

Another procedure sometimes recommended is to replace market salvage value with imputed salvage value and then proceed just as before. The imputed salvage value (ISV) can be viewed as the end-of-year 3 salvage value that would create a tie (between 3 years and 5 years in this case) for the economic life. To compute imputed salvage value, use the formula:

\[
\text{ISV}(n^*) = \left( p - s_n \delta^n \right) \left( \frac{1-\delta}{\delta(1-\delta^n)} \right) \left[ \frac{\delta(1-\delta^{n-n^*})}{(1-\delta)} \right] + s_n \delta^{n-n^*},
\]
where $\delta$ is the discount factor ($\delta = 1/(1+i)$, where $i$ is the interest rate), $n$ is the end of the economic life, $n^*$ is the length of the study period, and $s_n$ is the salvage value at the end of the asset's economic life (year $n$). This formula is equivalent to the expression found in [9]. If the interest rate is 10%, then the $ISV$ for alternative $A_1$ in our example is $\$10,783$, and the modified cash flows become:

<table>
<thead>
<tr>
<th>End of Year</th>
<th>$A_1$ Cash Flow</th>
<th>$A_2$ Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-20,000$</td>
<td>$-12,000$</td>
</tr>
<tr>
<td>1</td>
<td>$-5,000$</td>
<td>$-7,000$</td>
</tr>
<tr>
<td>2</td>
<td>$-5,000$</td>
<td>$-7,000$</td>
</tr>
<tr>
<td>3</td>
<td>$-5,000 + 10,783$</td>
<td>$-7,000 + 3,000$</td>
</tr>
</tbody>
</table>

Textbooks recommend the imputed salvage value procedure less frequently than the market salvage value method discussed previously. Further, textbook discussions of the imputed salvage value procedure (see, for example [9]) usually recommend that it be used only when market salvage values are unavailable. These discussions imply that market salvage values provide the correct economic measurement of end effects.

Our position in this paper is exactly opposite: imputed salvage value, not market salvage value, provides the correct economic measurement. One way of motivating our position, using the numerical example above, is to note that the alternative to replace after three years (using market salvage value) has already been dominated (during the determination of economic life) by the choice to replace in five years. Hence, valuing it with market salvage value erroneously burdens the asset with an already rejected scenario. To see why market salvage value is not necessarily an appropriate measure of economic value for used capital assets, it may be useful to consider, as an example, the market for used packaging machinery.

Packaging machinery, according to one of the leading manufacturers of packaging equipment, is usually custom designed and built to order. Thus, owners of used packaging machinery are usually unable to find a buyer who can use the equipment without substantial modifications. The market value of used, custom-built packaging machinery is, therefore, likely to be little more than its scrap value, even if the machinery is relatively new, up-to-date, and in excellent condition. Market salvage values of relatively new packaging machinery will clearly under-estimate the economic value to the firm of owning the equipment.

Another example in which the market is too "thin" to provide accurate economic valuation is the market for used telecommunications equipment. Telecommunications equipment typically becomes technologically obsolete long before reaching its service life.
Bergh, et al. [1] point out that the second-hand market for telecommunications equipment is almost non-existent, even though the equipment may still provide service.

It is important to note that in all of our discussion, we make the assumption that the services of the machine continue to be needed. We thus are not considering scenarios where the equipment, at an age prior to its economic life, is suddenly and unexpectedly no longer needed. Such scenarios certainly do occur when, for example, there is a factory shutdown or the enforcement of new government regulations; but when they do, there is no decision problem. The existing equipment must be sold at its market salvage value (which could be negative).

If we accept the argument that market salvage values may not provide correct economic measurements of end effects, then an important, unresolved problem is to determine a correct measure. The remainder of this paper provides our resolution. Section 2 shows how to generalize the concept of imputed salvage value, previously defined only for situations where operating and maintenance costs are age-independent, to the case of arbitrary operating and maintenance costs. In section 3, we use classical microeconomic theory to argue that our generalized imputed salvage values provide the correct economic measurements of end effects. We also show that the intuition derived from the packaging machinery example, that market salvage values underestimate economic value, is true in general. Section 4 summarizes the contributions of the paper and provides an example showing that the use of market salvage values to measure end effects rather than generalized imputed salvage values can lead to an incorrect capital investment decision.

2. GENERALIZED IMPUTED SALVAGE VALUES AND STATIONARY DUAL PRICES

a. Generalized Imputed Salvage Values

Let us review the concept of imputed salvage value (ISV) mentioned in the previous section (as introduced in [9]). Suppose an asset of age $j$ has maintenance and operating costs given by $m_j$. We will initially assume that:

$$m_j = m \text{ for } j = 1, \ldots, n.$$

where $n$ is the economic life of the asset. Let $p$ be the capital investment cost of the asset, and let $s_j (j = 1, \ldots, n)$ be the market salvage value of a $j$ year old asset. We assume that all assets provide the same service regardless of age, so revenues are not considered. The economic parameters depend only on the age of the asset, not on the given year.
The cash flow that represents an asset over its economic life is given by:

\[
\begin{array}{c|c}
\text{End of Year} & \text{Cash Flow} \\
0 & -p \\
1 & -m_1 \\
2 & -m_2 \\
\vdots & \vdots \\
\text{n} & -m_n + s_n
\end{array}
\]

The annual equivalent cost is given by:

\[
AE_n(\delta) = (p - s_n \delta^n) \frac{(1-\delta)}{\delta(1-\delta^n)} + m .
\]

To use the imputed salvage value approach when the study period, \(n^*\), is less than or equal to the economic life, \(n\), we compute \(ISV(n^*)\) and modify the above cash flow as discussed in the introduction. First:

\[
ISV(n^*) = \left[ (p - s_n \delta^n) \frac{(1-\delta)}{\delta(1-\delta^n)} \right] \left[ \frac{\delta(1-\delta^{n-n^*})}{(1-\delta)} \right] + s_n \delta^{n-n^*}.
\]

Next, the modified cash flow is given by:

\[
\begin{array}{c|c}
\text{End of Year} & \text{Modified Cash Flow} \\
0 & -p \\
1 & -m_1 \\
2 & -m_2 \\
\vdots & \vdots \\
\text{n} & \vdots \\
\text{n^*} & -m_{n^*} + ISV(n^*)
\end{array}
\]

The imputed salvage value, \(ISV(n^*)\), is the value that equates annual equivalent cost of the modified cash flow stream to annual equivalent cost of the original cash flow stream. To see this, let \(AE_{n^*}(\delta)\) denote the annual equivalent cost of the modified cash flow stream.

It is then the case that:

\[
AE_{n^*}(\delta) = \left[ p - ISV(n^*) \delta^{n^*} \right] \frac{(1-\delta)}{\delta(1-\delta^{n^*})} + m .
\]
Substituting in the formula for $ISV(n^*)$, it follows from simple algebra that $AE_n^*(\delta) = AE_n(\delta)$. Thus, as we claimed, $ISV(n^*)$ can be viewed as the number that equates annual equivalent costs between the original and the modified cash flow streams.

Note that the maintenance term $m$ falls out of this calculation because it is a common term. Although the concept of imputed salvage value was developed for the case of age-independent maintenance costs only, the concept also applies to the more general case where maintenance costs are age-dependent. To see why, suppose the $m_j$ coefficients are not necessarily constant for $j=1,...,n$. Let $GISV(n^*)$ (where $GISV$ stands for generalized imputed salvage value) be defined as the number that equates annual equivalent costs of the modified and original cash flow streams when $GISV(n^*)$ replaces $s_n^*$ in the original cash flow stream. The formulas for $AE_n(\delta)$ and $AE_n^*(\delta)$ are now more complicated expressions involving $m_j (j=1,...,n)$:

$$AE_n(\delta) = (p - s_n \delta^n) \frac{(1-\delta)}{(1-\delta^n)} + \left[ \sum_{j=1}^{n} m_j \delta^j \right] \frac{(1-\delta)}{(1-\delta^n)}, \text{ and}$$

$$AE_n^*(\delta) = \left[ p - GISV(n^*) \delta^{n^*} \right] \frac{(1-\delta)}{(1-\delta^{n^*})} + \left[ \sum_{j=1}^{n^*} m_j \delta^j \right] \frac{(1-\delta)}{(1-\delta^{n^*})}.$$

$GISV(n^*)$, by definition, must satisfy $AE_n(\delta) = AE_n^*(\delta)$. If $n^* = n$, obviously $GISV(n) = s_n$. Otherwise, if $n^* = 1,...,n-1$, solving $AE_n(\delta) = AE_n^*(\delta)$ for $GISV(n^*)$, yields:

Claim 1:

$$GISV(n^*) = AE_n(\delta) \frac{\delta^{(1-\delta^{n-n^*})}}{(1-\delta)} + s_n \delta^{n-n^*} - \sum_{j=n^*+1}^{n} m_j \delta^{j-n^*}.$$

Derivation: See the appendix for details.

As an example, consider the following modifications of cash flows $A_1$ and $A_2$ where maintenance costs are not constant over the economic lives of the assets.
<table>
<thead>
<tr>
<th>End of Year</th>
<th>A_3 Cash Flow</th>
<th>A_4 Cash Flow</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>-20,000</td>
<td>-12,000</td>
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</tr>
<tr>
<td>5</td>
<td>-9,000 + 3,000</td>
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</tbody>
</table>

Again, suppose that the market salvage value of asset A_3 at the end of three years is $6,000, and that the interest rate is 10%. Computing GISV(3) for asset A_3 from our formula, we obtain:

$$GISV(3) = 7,892.$$  

Market salvage value and GISV have different values and, therefore, can have different impacts on decision making.

We now show that GISV values are the same as the stationary dual prices discussed by the authors in [6] and [7], and that they measure the economic value of owning the capital asset.

b. Stationary Dual Prices

In [6], we developed a theory of economic evaluation for the simple capital investment model considered here. We used infinite-horizon linear programming and duality theory to demonstrate the existence of stationary dual prices that measure the economic value of owning a capital asset. In summary, the stationary dual price of a j year old asset is denoted by $\tilde{y}(j)$ and defined as:

$$\tilde{y}(j) = \begin{cases} 
\tilde{y} - \delta n_{j+1} + \delta \tilde{y}(j+1), & j=1,2,\ldots,n-1 \\
\tilde{y}, & j=n
\end{cases}$$

where $\tilde{y} = AE_n(\delta)$. It follows from a recursive argument that $\tilde{y}(j)$ is the economic value of a capital asset of age j. To see why, first note that the annual equivalent cost of the services provided by the capital asset and hence, (in a profit-maximizing industry where marginal cost and marginal revenue are equated) the annual economic value of the services provided by the machine is given by $\tilde{y}$. Clearly, at the end of its economic life, the
economic value of the asset is its market salvage value, $s_n$. The economic value of an asset which has not yet reached its economic life must equal the annual value of the services it provides, less its discounted operating and maintenance costs, plus its discounted economic value one year hence. Thus, working back from the economic life, the stationary dual prices measure the economic value of the asset at each different age.

The definition of $\bar{y}$ provided above is recursive. More explicitly, for $n^* = 1, \ldots, n-1$, we can show:

Claim 2:

$$\bar{y}(n^*) = AE_n(\delta) \frac{\delta(1-\delta^{n-n^*})}{(1-\delta)} + s_n \delta^{n-n^*} - \sum_{j=n^*+1}^{n} m_j \delta^{j-n^*}.$$ 

Derivation: See the appendix for details.

This, however, is just the formula for $GISV(n^*)$, where $n^*$ is less than $n$. Since $GISV(n) = s_n$, as does $\bar{y}(n)$, we conclude that generalized imputed salvage values and stationary dual prices are the same. Thus, $GISV(n^*)$ measures the economic value of an $n^*$ year old capital asset. Moreover, in the next section, we use microeconomic theory to argue that $GISV$ values provide correct economic measurement of end effects, i.e., the effects of retiring the asset at an age other than its economic life.

3. MEASURING END EFFECTS WITH $GISV$

a. Literature of Economic Evaluation of End Effects

To successfully argue for the correct way to measure the end effects of retiring capital assets at ages other than their economic lives, one must first decide how to correctly determine the economic lives. Henderson and Quandt [3], for instance, make the point that determining how long to keep a single machine is an inherently different problem than determining how long to keep a machine that will be replaced by a like machine. As they observe, the basic question in determining the economic life of a capital asset is whether one should optimize over one cycle (a finite-horizon problem) or over all future cycles (an infinite-horizon problem). Hotelling [4], who studied the depreciation of vintage machinery by optimizing over one replacement cycle only, ignored this basic question.

The same basic question arises in other contexts. In forestry economics, for example, a debate that lingered for years was whether to optimize over a single rotation or over all future rotations when determining the length of the optimal rotation. See [5] and
for discussions of this longstanding and controversial question. Samuelson finally resolved this question in a classic paper [8] which shows that the correct economic life or rotation cycle should be found by optimizing over all future cycles. Samuelson also pointed out that you can correctly optimize over a finite-horizon, but to do so, you must use the proper economic evaluation of the worth of the asset (in his context, land rent) to handle the end effects. He further demonstrated that length of the cycle determined by optimizing over a finite-horizon will be longer than it should be, unless end effects are properly handled. We now extend Samuelson's arguments to the context of capital investment.

b. G1SV Correctly Evaluates End Effects

We first observe that decisions about operating capital assets are a sequence of one period decisions. If the services of an asset are required in any year, then the current asset must either be kept or replaced. Thus, we base our analysis on a one-year study period in which we can either keep or replace an asset. We will show that the one-year study period can prejudice the current decision if we do not use the appropriate measure of economic end effects, i.e., G1SV.

Before showing that G1SV values correctly evaluate end effects, we first demonstrate that market salvage values do not. The telecommunications and packaging equipment examples in the introduction provide an intuitive rationale for believing that market salvage values may not correctly evaluate the economic worth of used assets. The example that follows provides a concrete confirmation of this intuition.

Using market salvage values to evaluate the worth of used assets, the current decision for a \( j \)-year-old asset is represented by:

\[
\text{max} \begin{cases} 
\text{keep:} & - \delta m_{j+1} + \delta s_{j+1} \\
\text{replace:} & s_j - p - \delta m_1 + \delta s_1 
\end{cases}
\]

For \( j=1,\ldots,n-1 \), we know that the correct decision is to keep the asset. Only at \( j=n \) should the asset be replaced. However, decisions arrived at by solving the above optimization problem are not necessarily correct. For instance, consider the asset represented by cash flow \( A_3 \). Let the market salvage values for this asset over its economic life (5) be given by:

\[
s_1 = 15,000, \quad s_2 = 10,000, \quad s_3 = 6,000, \quad s_4 = 4,000, \quad s_5 = 3,000.
\]

Again let \( i = 10\% \). For \( j=1 \), the value of the above replace option is $4,091, and the value of the above keep option is $3,636. Hence, the decision based on this criteria is to replace
the one-year old asset. This decision is incorrect because the asset has not yet reached its economic life of 5 years.

Suppose the sequential economic life calculation for this example is repeated, but now using $GISV$ values in place of market salvage values. It turns out that the decision will be to keep the machine until it is 5 years old and then to replace it. The reason that using market salvage values produces an incorrect answer, in this example, at least, is because market salvage values under-estimate economic worth. The following theorem shows that this conclusion is true in general.

**Theorem 1**: $GISV(n^*) \geq s_{n^*}$ for $n^* = 1, 2, \ldots, n$.

**Proof**: If an asset is sold when it is $n^*$ years old for the salvage value $s_{n^*}$, where $1 \leq n^* \leq n$, the annual equivalent cost of the corresponding cash flow, denoted by $AE_{n^*}(\delta)'$, is:

$$AE_{n^*}(\delta)' = (p - s_{n^*} \delta^{n^*}) \frac{(1-\delta)}{\delta(1-\delta^{n^*})} + \left[ \sum_{j=1}^{n^*} m_j \delta^j \right] \frac{(1-\delta)}{\delta(1-\delta^{n^*})}.$$

By definition of the economic life ($n$), $AE_n(\delta) \leq AE_{n^*}(\delta)'$. By definition of $GISV(n^*)$, $AE_{n^*}(\delta) = AE_{n^*}(\delta)$. Therefore, $AE_{n^*}(\delta) \leq AE_{n^*}(\delta)'$. This inequality implies that, $p - GISV(n^*) \delta^{n^*} \leq p - s_{n^*} \delta^{n^*}$. Simplifying this inequality yields, $s_{n^*} \leq GISV(n^*)$.

To see that $GISV$ values correctly measure economic end effects, suppose we measure the worth of the services of the asset (or an equivalent but younger asset if it is replaced) in the next year using its $GISV$. The following optimization problem represents the current decision:

$$\max \left\{ \begin{array}{l}
\text{keep: } -\delta m_{j+1} + \delta GSV(j+1) \\
\text{replace: } s_j - p - \delta m_1 + \delta GSV(1)
\end{array} \right.$$

**Theorem 2**: $GISV$ correctly evaluates end effects.

**Proof**: For $j=1, \ldots, n-1$:

$$-\delta m_{j+1} + \delta GSV(j+1) = GSV(j) - \bar{y}, \text{ and}$$

$$s_j - p - \delta m_1 + \delta GSV(1) = s_j - \bar{y}.$$
From Theorem 1, \( GISV(j) \geq s_j \). Hence, the value of the keep option is always at least as large as the value of the replace option for \( j=1,\ldots,n-1 \). Now, suppose \( j=n \). It follows that:

\[
\tilde{y}(n) - \tilde{y} - \delta \tilde{y}(n+1) \geq -\delta m_{n+1}, \text{ and}
\]

\[
\tilde{y} = p + \delta m_1 - \delta \tilde{y}(1).
\]

It then follows that:

\[
-\delta m_{n+1} + \delta GISV(n+1) \leq s_n - p - \delta m_1 + \delta GISV(1).
\]

Hence, the value of the replace option is at least as large as the value of the keep option at the economic life. Thus, the correct decision is always made when \( GISV \) is used to evaluate end effects.

In the numerical example preceding the two theorems, using market salvage values to evaluate end effects led to replacement before the economic life. It is interesting to note that using market salvage values to evaluate end effects often yields an incorrect decision to keep longer than the economic life rather than to replace early, as in the example. To see why, recall that to determine the economic life of an asset, you must minimize the following function (corresponding to the present worth of an infinite stream of like replacements) over \( k=1,2,\ldots \):

\[
p + \sum_{j=1}^{k} \frac{\delta^j m_j - \delta^j s_j}{1 - \delta^k}.
\]

When this function is convex, it follows from some algebraic manipulation that using market salvage values to evaluate end effects will result in keeping the asset at least as long as the economic life (in general, longer). This observation corresponds exactly to Samuelson's observation [8] that optimizing the economic yield of the forest over just one replacement cycle (if market values are used) yields a rotation cycle (age at which the trees are harvested) that is too long.
4. CONCLUSIONS

This paper has studied the problem of evaluating end effects with imputed salvage values when the study period is shorter than the asset's economic life. We have generalized the concept of imputed salvage value, normally defined only for assets whose maintenance and operation costs are independent of age, to include assets with arbitrary operating and maintenance costs. We also have shown that generalized imputed salvage values are equivalent to the stationary dual prices of an infinite-horizon linear program. Finally, we have argued that imputed salvage values (not market salvage values) correctly measure economic value and, hence, appropriately evaluate end effects.

As a final point, we present an example that shows how using market salvage values to evaluate end effects can yield incorrect capital investment decisions. To do so, let us return to the study period problem (our original motivation) with the two alternatives, A₃ and A₄. We will show that using the market salvage value approach yields a different (and, hence, incorrect) decision than we get using the GISV approach. For a study period of 3 years, we must truncate cash flow A₃ so as to fairly compare assets A₃ and A₄. This requires that we evaluate the effects of truncating cash flow A₃ by including a receipt at year 3 on the modified cash flow A₃. We have just argued that GISV(3) = $7,892 is the correct receipt, not the market salvage value (MSV) of $6,000. Let us analyze the effect of this conclusion on the ultimate decision: whether to choose asset A₃ or A₄. The cash flows, assuming an interest rate, i, of 10%, are presented in the following table.

<table>
<thead>
<tr>
<th>Year</th>
<th>Modified A₃ (MSV)</th>
<th>Modified A₃ (GISV)</th>
<th>A₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-20,000</td>
<td>-20,000</td>
<td>-12,000</td>
</tr>
<tr>
<td>1</td>
<td>-5,000</td>
<td>-5,000</td>
<td>-7,000</td>
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<td>2</td>
<td>-6,000</td>
<td>-6,000</td>
<td>-8,000</td>
</tr>
<tr>
<td>3</td>
<td>-7,000 + 6,000</td>
<td>-7,000 + 7,892</td>
<td>-9,000 + 3,000</td>
</tr>
</tbody>
</table>

The present values of the three cash flow streams, again assuming an interest rate of 10%, are shown below.

<table>
<thead>
<tr>
<th>Modified A₃ (MSV)</th>
<th>Modified A₃ (GISV)</th>
<th>A₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>-30,255</td>
<td>-28,834</td>
<td>-29,483</td>
</tr>
</tbody>
</table>

Clearly the use of GISV rather than MSV to evaluate end effects has a critical impact on decision-making. If we had used the market salvage value of $6,000 as the
terminal receipt for asset A₄, then we would choose A₄ over A₃ because A₄ has the smaller present worth costs. However, using the GISV of $7,892 as the terminal receipt for asset A₃ leads us to the correct decision: choose A₃ over A₄. Thus, improper evaluation of end effects in study period problems can lead to incorrect decision-making, a problem that can be avoided by using GISV.

APPENDIX

Claim 1: For n*, 1,...,n-1, GISV(n*) is given by:

\[ GISV(n^*) = AE_\frac{n}{n} (\delta) \frac{\delta(1-\delta^{n-n^*})}{(1-\delta)} + s_n \delta^{n-n^*} - \frac{n}{\sum_{j=n^*+1} m_j \delta^{j-n^*}.} \]

Proof: By definition, GISV(n*) must satisfy \( AE_\frac{n}{n} (\delta) = AE_\frac{n}{n^*} (\delta). \) This equation implies that:

\[ (p - s_n \delta^n) \frac{(1-\delta)}{\delta(1-\delta^n)} + \left[ \sum_{j=1}^{n^*} m_j \delta^j \right] \frac{(1-\delta)}{\delta(1-\delta^n)} = \]

\[ [p - GISV(n^*) \delta^{n^*} + \frac{(1-\delta)}{\delta(1-\delta^{n^*})} \left[ \sum_{j=1}^{n^*} m_j \delta^j \right] \frac{(1-\delta)}{\delta(1-\delta^{n^*})}. \]

Solving this equation for GISV(n*), it follows that:

\[ GISV(n^*) = p \left[ \frac{1-\delta^{n-n^*}}{1-\delta^n} \right] + (\delta m_1 + ... + \delta^{n^*} m_{n^*}) \left[ \frac{1-\delta^{n-n^*}}{1-\delta^n} \right] \]

\[ - (\delta^{n^*+1} m_{n^*+1} + ... + \delta^n m_n) \left[ \frac{1-\delta^{n^*}}{\delta^{n^*} (1-\delta^n)} \right] \]

\[ + s_n \delta^{n-n^*} \left[ \frac{(1-\delta^{n^*})}{(1-\delta^n)} \right]. \]

(1)

It is also true that:
\[(\delta m_1 + \ldots + \delta^m m_n) = (\delta m_1 + \ldots + \delta^m m_n) \]

\[-(\delta^{m+1} m_{m+1} + \ldots + \delta^m m_n) \quad (2)\]

and
\[s_n \delta^{n-n} \left[ \frac{(1-\delta^n)}{(1-\delta)} \right] = s_n \delta^{n-n} - s_n \frac{\delta^n}{(1-\delta^n)} \cdot \quad (3)\]

Substituting (2) and (3) into (1), it follows that:

\[GISV(n^*) = \left[ (p - s_n \delta^n) + \sum_{j=1}^{n} m_j \delta^j \right] \left[ \frac{(1-\delta^n)}{(1-\delta^n)} \right] \left[ \frac{\delta(1-\delta^{n-n})}{\delta^n} \right] + s_n \delta^{n-n} \]

\[- \delta^{n^*} (\delta m_{n+1} + \ldots + \delta^{n-n} m_n) \left[ \frac{(1-\delta^{n-n})}{(1-\delta^n)} + \frac{(1-\delta^n)}{\delta^n (1-\delta^n)} \right]. \quad (4)\]

However,
\[\frac{(1-\delta^{n-n})}{(1-\delta^n)} + \frac{(1-\delta^n)}{\delta^n (1-\delta^n)} = \frac{1}{\delta^{n^*}}, \quad (5)\]

and,

\[AE_n(\delta) = \left[ (p - s_n \delta^n) + \sum_{j=1}^{n} m_j \delta^j \right] \left[ \frac{(1-\delta)}{\delta(1-\delta^n)} \right]. \quad (6)\]

Finally, if we substitute (5) and (6) into (4), we have the desired result.

**Claim 2:** \(GISV(n^*) = \bar{y}(n^*), \) for \(n^* = 1, \ldots, n.\)

**Proof:** The case for \(n^* = n\) is trivial since both \(GISV(n)\) and \(\bar{y}(n)\) are \(s_n\). Next, let \(1 \leq n^* \leq n-1.\) The stationary dual prices \(\bar{y}(n^*)\) are defined recursively by:
\[ \gamma(n^*) = \gamma - \delta m_{n^*+1} + \delta \gamma(n^*+1), \]

where this recursion is completely defined by \( \gamma(n) = s_n \). If we evaluate this recursion by back substitution, we find that:

\[ \gamma(n^*) = \gamma(1) \frac{(1-\delta^{n-n^*})}{(1-\delta)} + s_{n^*} \delta^{n-n^*} - \sum_{j=n^*+1}^{n} m_j \delta^{j-n^*}. \]  \hspace{1cm} (7)

However \( \gamma = AE_n(\delta) \). If we substitute this expression into (7), we have precisely the expression for \( GISV(n^*) \).

REFERENCES


