Machine Maintenance with Multiple Maintenance Actions

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Abstract: A maintenance model under Markovian deterioration is developed in which maintenance and replacement actions are permitted and states are completely observable. The optimal policy for the case where the effect of maintenance actions is stochastic is shown to have a monotonic control limit structure. The special case where maintenance actions are deterministic is also discussed and the effect of alternate modeling assumptions is considered.

Considerable effort has been devoted to modeling the economic decision problem posed by the need to replace equipment subject to deterioration. Methods of engineering economics (Terborgh [17]), Markov decision processes (Anderson [1], Derman [4]), dynamic programming (Eppen [6], optimal control theory (Thompson [18], Kamien and Schwartz [10]), partially observable Markov decision processes (Eckles [5]) and many variations of these have been applied to this problem. Surveys of the maintenance/replacement literature include Barlow, Proschan and Hunter [2], McCall [12], Pierskalla and Voelker [13], and Sherif and Smith [16]. In the vast majority of these models, the only maintenance action available is “replacement,” possibly supplemented by “inspection” in models where the state of the equipment is unobservable. When “repair” is listed as an action, it is most often equivalent to “replacement,” since the system is assumed to be returned to a state as good as new.

In many applications, ranging from automobile maintenance to file management in a computer system, however, intermediate maintenance actions are available to improve the state of the system without returning it to its new state. In these problems, the issue is not simply when to replace the equipment, but instead which maintenance actions to choose and when to implement them. This paper addresses the issue of modeling this type of multiple action maintenance model where the states are one dimensional and completely observable.

Our main results show that, under reasonable conditions, the optimal maintenance policy will (1) have a control limit structure (i.e., there exists a deterioration level below which it is optimal to “do nothing,” and above which it is optimal to perform some type of maintenance on the system) and (2) be monotonic (i.e., the optimal maintenance action will either increase or decrease, depending on the specific assumptions, in the level of deterioration, where maintenance actions are labeled in order of decreasing effectiveness). These results give intuitive insight into the behavior of multiple action maintenance problems and could also serve as the basis for simplifying the computation of optimal policies.

Model Description

We consider a machine that is observed at discrete intervals on a state space $S = \{1, \ldots, n\}$, where states are numbered such that 1 represents the best state and n represents the worst state. If the machine is in state $i$ at any observation epoch, the decision-maker has the option of implementing a maintenance action from the set $X_i = \{m_i, \ldots, N\}$, where the actions are numbered such that the effectiveness of the actions decrease with their index number. We will make this notion of decreasing effectiveness more precise in the next section. In particular, action $N$ signifies the “do nothing” action. We assume that $m_i \leq m_j$ for $i < j$ (i.e., $X_i \supseteq X_j$), so that all maintenance actions in state $j$ are available in a better state $i$. Taking action $a \in X_i$ in state $i$ at the beginning of a period results in an immediate reward of $R(i,a)$ and causes the system to be in state $j$ at the beginning of the next period with probability $p_{ij}(a)$. 

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The problem facing the decision-maker is to choose a sequence of maintenance actions so as to maximize the \( \beta \)-discounted total net present reward over the infinite horizon. We assume \( \beta < 1 \). The optimal net present value of owning a machine in state \( i \) is denoted by \( f_i \), and satisfies the following dynamic programming equation (DPE):

\[
f_i = \max_{a \in X_i} \{ R(i,a) + \beta \sum_{j=1}^{n} p_{ij}(a)f_j \}, \quad i \in S.
\] (1)

It is well-known by the theory of Markov decision processes that the value function exists, is bounded, and is achieved by a stationary policy of the form \( x^* = [x_i^*] \), \( i = 1, \ldots, n \), where \( x^*_i \) represents the optimal action in state \( i \) at any point in time. This solution can be computed by using the familiar policy improvement and value iteration methods (Howard [9], Denardo [3]). To break ties, we define \( x^*_i \) to be the largest element in \( X_i \) that achieves the max in DPE (1).

**Assumptions**

To further characterize the structure of the optimal stationary policy, \( x^* \), we will make use of the following assumptions.

(A1) \( \sum_{j=1}^{k} p_{ij}(a) \geq \sum_{j=1}^{k} p_{ij}(\bar{a}) \) for any \( \bar{a} \leq \bar{a} \), \( i \leq m \), \( k \in S \).

Assumption (1) is a broader version of the commonly used IFR (increasing failure rate) assumption used to model Markovian deterioration (Derman [4], Ross [14]). If \( a = \bar{a} = N \), then assumption (1) is precisely the usual IFR assumption. If \( a < \bar{a} \) and/or \( i < m \), then it implies that worse initial states and/or less effective actions lead to stochastically worse states after the next transition.

(A2) \( R(i,a) \) is nonincreasing in \( i \) for any fixed \( a \in X_i \), and nonincreasing in \( a \) for any fixed \( i \in S \).

Assumption (2) implies that states and actions with higher indices (i.e., worse states and less effective actions) produce smaller one period rewards.

(A3) \( R(k,a) - R(k,N) \geq R(i,a) - R(i,N) \) for any \( k > i \), \( a \in X_i \).

Assumption (3) means that the difference between performing maintenance and doing nothing is larger for a worse state than a better state with respect to the effect on the expected reward in the current period.

(A4) \( \sum_{j=1}^{k} (p_{ij}(a) - p_{ij}(N)) \) is nondecreasing in \( i \) for all \( k \in S \) and \( a \in X_i \).

Assumption (4) is the counterpart to assumption (3) and implies that the difference between performing maintenance and doing nothing is larger for a worse state than a better state with respect to their effect on the probability distribution in the next period.

Lastly, we state two assumptions with an "either-or" structure. If assumptions (5a) and (6a) hold, we will get one type of structural result for the optimal policy. If, instead, assumptions (5b) and (6b) hold, we will get the opposite type of structure.

(A5) a) \( R(i,\bar{a}) - R(i,\bar{a}) \) is nonincreasing in \( i \) for \( \bar{a} \leq \bar{a} < N \),

b) \( R(i,\bar{a}) - R(i,\bar{a}) \) is nondecreasing in \( i \) for \( \bar{a} \leq \bar{a} < N \),

Assumption (5a) requires reward functions to have antitone differences, while (5b) requires isotope differences (Anderson [1]). Notice that it is possible for the reward functions to have both isotope and antitone differences if these differences are constant with respect to \( i \).

(A6) a) \( \sum_{j=1}^{k} (p_{ij}(\bar{a}) - p_{ij}(\bar{a})) \) is nonincreasing in \( i \) for \( \bar{a} \leq \bar{a} < N \), \( k \in S \),

b) \( \sum_{j=1}^{k} (p_{ij}(\bar{a}) - p_{ij}(\bar{a})) \) is nondecreasing in \( i \) for \( \bar{a} \leq \bar{a} < N \), \( k \in S \).

Assumptions (6a) are of the same type as assumption (4). Hence, (6a) states that improved maintenance makes a larger difference on a better state than a worse state; conversely, assumption (6b) states that improved maintenance makes a smaller difference on a better state than a worse state. As with assumptions (5), it is possible for assumptions (6a) and (6b) to be satisfied simultaneously, provided that the indicated differences are constant in \( i \).

**Structural Results**

We now use the above assumptions to develop structural results for the optimal policy. To do this, we will make use of Derman's lemma (Derman [4]), which can be stated as follows.

**Lemma 1 (Derman):** For an \((n \times n)\) stochastic matrix \( P = [p_{ij}] \), the following are equivalent:

a) \( \sum_{j=1}^{k} p_{ij} \) is nonincreasing in \( i \) for \( k = 1, \ldots, n \) (i.e., \( P \) is IFR)
b) $\sum_{j=1}^{x} p_{ij} g(j)$ is nonincreasing in $i$ for any bounded non-increasing function $g$.

With this lemma and assumptions (1) and (2) we can demonstrate the intuitive result that the value function is monotonically decreasing in the state.

**Lemma 2:** Under assumptions (1) and (2), $f_{i}$ is nonincreasing in $i$.

**Proof:** See Appendix.

To demonstrate our major results, we will also need the following variation of Derman's lemma.

**Lemma 3:** If $\tilde{a} \leq a$ then the following conditions are equivalent:

a) $\sum_{j=1}^{x} (p_{ij}(\tilde{a}) - p_{ij}(a))$ is nonincreasing (nondecreasing) in $i$ for $k=1,\ldots,n$.

b) $\sum_{j=1}^{x} (p_{ij}(\tilde{a}) - p_{ij}(a)) g(j)$ is nonincreasing (nondecreasing) in $i$ for any bounded nonincreasing function $g$.

**Proof:** See Appendix.

Theorems 1 and 2 state the main results of this section by giving conditions under which the optimal policy will be a control limit policy and monotonic.

**Theorem 1:** Under assumptions (1)-(4), the optimal policy has the property that for some $i^{*}$, $x_{i}^{*} = N$ for $i < i^{*}$ and $x_{i}^{*} < N$ for $i \geq i^{*}$.

**Proof:** See Appendix.

**Theorem 2:**

a) Under assumptions (1)-(4), (5a) and (6a), $x_{i}^{*}$ is non-decreasing in $i$ for $i \geq i^{*}$.

b) Under assumptions (1)-(4), (5b) and (6b), $x_{i}^{*}$ is non-increasing in $i$ for $i \geq i^{*}$.

**Proof:** See Appendix.

The general conclusion embodied in Theorems 1 and 2 is that it is optimal to initially allow a new machine to deteriorate without interruption up to some control limit $i^{*}$ and then apply either increasingly or decreasingly costly maintenance as the machine reaches progressively worse states. Whether the maintenance levels increase or decrease with the state above the control limit depends on whether assumptions (5a) and (6a) or assumptions (5a) and (6b) hold. If both sets of assumptions hold, then Theorem 2 implies that actions above the control limit are constant. We consider some special cases that also lead to a constant maintenance control limit policy in the following section.

We conclude this section by observing that if assumptions (3) and (4) are altered appropriately, then the conclusion of Theorem 1 is reversed. The necessary variants of these assumptions are:

(A3') $R(k,a) - R(k,N) \leq R(i,a) - R(i,N)$ for any $k > i$, $a \in \mathcal{X}_{i}$.

(A4') $\sum_{j=1}^{x} (p_{ij}(a) - p_{ij}(N))$ is nonincreasing in $i$ for all $k \in \mathcal{S}$.

With these modified assumptions, we can prove the following control limit result.

**Theorem 3:** Under assumptions (1), (2), (3'), and (4'), the optimal policy has the property that for some $i^{*}$, $x_{i}^{*} < N$ for $i < i^{*}$ and $x_{i}^{*} = N$ for $i > i^{*}$.

**Proof:** The proof is identical to that of Theorem 1. ■

The same reasoning used in Theorem 2 could be applied to this situation to show that, below the control limit $i^{*}$, the maintenance actions are monotonically increasing in $i$ if assumptions (5a) and (6a) hold and monotonically decreasing in $i$ if assumptions (5b) and (6b) hold. Hence, the optimal policy under these conditions consists of applying increasingly or decreasingly costly maintenance to the system until it reaches some control limit, beyond which it is optimal to abandon it.

**Extensions and Special Cases**

By making additional assumptions about the effect of maintenance actions we can characterize the optimal maintenance policy even more precisely. We show that a simple "constant maintenance control limit" structure results if reward functions are separable and either the effect of maintenance actions is assumed independent of the current state or maintenance actions are assumed deterministic. In addition, we show that the structural conclusions of this paper are partially robust under alternate modeling assumptions concerning the deterioration and maintenance processes.

**State Independent Maintenance**

Suppose that the transition probabilities resulting from an action other than "do nothing" do not depend on the curr-
rent state. This might be the case if the maintenance action involves replacement of certain components so that the condition of the equipment after maintenance depends only on the quality of the new components and not on the old ones being replaced. We can express this assumption as:

(A7) \( R(i, a) = r(i) - c(a), \) where \( r(i) \) is nondecreasing in \( i \) and \( c(a) \) is nonincreasing in \( a \).

(A8) a) \( X_i = X \) for all \( i = 1, ..., n \).

b) For each \( a \in X \) there exists \( q_j(a), j = 1, ..., n \) such that \( p_{j0}(a) = q_j(a) \) for all \( i = 1, ..., n \).

We now show that these assumptions result in a constant maintenance control limit policy.

**Theorem 4:** Under assumptions (1), (7) and (8), there exist \( i^* \) and \( a^* \) such that \( x^* = N \) for \( i < i^* \) and \( x^* = a^* \) for \( i \geq i^* \).

**Proof:** It is straightforward to show that assumptions (7) and (8) imply assumptions (2), (3), (4), (5) and (6). Hence, the result follows from Theorem 2. ■

Under the constant maintenance control limit policy in Theorem 4, optimal maintenance will simply consist of waiting for the system to reach or pass a control limit \( i^* \) and then performing action \( a^* \).

**Deterministic Maintenance**

Suppose now that the effect of maintenance actions is completely deterministic. That is, the state the system will be in after maintenance is known with certainty before the maintenance action is taken. In this case, maintenance actions can be identified by the state to which they move the system. If two actions move the system to the same state starting from some state \( i \), then clearly the more expensive action can be excluded without loss of optimality. Hence, we can express the action space in state \( i \) as \( X_i \subseteq S \setminus \{N\} \). \( X_i \) may not include all of \( S \) if some states are unreachable from \( i \), but we assume that action \( N \), the "do nothing" action is feasible in all states. To be consistent with our earlier formulation, we continue to assume that \( X_i \supseteq X_j \) for \( i < j \). Notice that under these conditions, action \( a \) in state \( i \) amounts to preventive maintenance, since this action prevents the system from leaving state \( i \). Depending on the particular application such preventive maintenance may or may not be feasible.

We state the deterministic maintenance case with two assumptions, the first being the usual IFR Markovian deterioration assumption and the second being the actual deterministic maintenance assumption.

(A9) \( \sum_{j=1}^{k} p_{j0}(N) \) is nonincreasing in \( i \) for \( k = 1, ..., n \).

(A10) For \( a \in X_i \setminus \{N\} \)

\[
p_{i0}(a) = \begin{cases} 1, & j = a \\ 0, & \text{otherwise} \end{cases}
\]

We can use these assumptions along with the separable rewards assumption (7) to prove that the optimal policy for this problem also has a constant maintenance control limit structure.

**Theorem 5:** Under assumptions (7), (9) and (10), there exist \( i^* \) and \( a^* \) such that \( x^* = N \) for \( i < i^* \) and \( x^* = a^* \) for \( i \geq i^* \).

**Proof:** We have already shown that assumption (7) implies assumptions (2), (3), (5a) and (5b). It is also easy to show that assumptions (9) and (10) imply assumption (4) and assumption (10) implies assumptions (6a) and (6b). The only assumption required in Theorem 2 that is not satisfied is assumption (1). The only need for this assumption is in Lemma 2 to show that \( f_i \) is nonincreasing in \( i \), which can be shown directly for the deterministic maintenance case by using a proof similar to that of Lemma 2. ■

Theorems 4 and 5 show that under the assumption of separable rewards, state independent maintenance and deterministic maintenance both lead to the same simple constant maintenance control limit structure of the optimal policy.

**Alternate Modeling Assumptions**

The above results are premised on a maintenance model in which maintenance actions are chosen (and paid for) at the beginning of the period, the system continues operating in its present state (and earning reward) during the period, and makes a transition to a new state at the end of the period. The question arises, therefore, to what extent the structural conclusions depend on these specific assumptions. We show below that these results are partially robust by considering two alternate maintenance models.

First, we consider the case where the system must be shut down for the entire period in order to perform maintenance. Thus, whenever a maintenance action other than "do nothing" is taken, no reward is earned during the period. Hence we can write \( R(i, N) = r(i) \) and \( R(i, a) = -c(a) \) for \( a < N \). Under these conditions, DPE (1) can be expressed as:

\[
f_i = \max \left\{ \begin{array}{l} r(i) + \beta \sum_{j=i}^{n} p_{j0}(N) f_j \\ \max_{a \in X_i \setminus \{N\}} -c(a) + \beta \sum_{j=i}^{n} p_{j0}(a) f_j \end{array} \right\}
\]

(2)

It is a straightforward matter to show that the proofs...
Lemma 2 and Theorems 1 through 5 go through directly with this DPE. Hence, structural conclusions are unchanged.

Second, we consider the case where deterministic maintenance takes place in so short an amount of time relative to the length of a period that it can be regarded as instantaneous. In this case, an action taken at the beginning of a period immediately moves the system to a new state, so that the reward earned during the period is a function of the new state and not the original state. Under these conditions, the DPE to compute the optimal policy becomes:

$$f_i = \max \left\{ r(i) + \beta \sum_{j=1}^{n} p_{ij}(N) f_j - c(a) + \sum_{j=1}^{n} p_{ij}(a) f_j \right\}$$

(3)

The only difference between DPE (3) and DPE (4) is the absence of the discount factor $\beta$ in the terms for actions $a < N$. While this does not affect the proof of Lemma 2, it does cause problems in the proof of Theorem 1. Since a control limit need not exist, Theorem 2 also does not hold. However, a modified version of the proof of Theorem 2, which shows that if $m > i$, $x_i < N$ and $x < N$ then, under assumptions (5a) and (6a), $x_i \geq x^*$, and under assumptions (5b) and (6b), $x_i \leq x^*$ still holds. Hence, we could characterize the optimal policy for the instantaneous maintenance case under assumptions (1)-(4) and either (5a) and (6a) or (5b) and (6b) as having a monotonic structure except for "do nothing" actions.

Applications

To validate our assumptions we consider some specific maintenance problems. First of all, we consider the case where when action $a$ is taken in some state $i$ the system either remains in state $i$ with probability $1 - \alpha$ or moves to state $a$ with probability $\alpha$. If the "do nothing" action is taken, then the system deteriorates according to an upper triangular IFR transition matrix (i.e., the $p_{ij}(N)$ are assumed to satisfy (A2)) that has $p_{ii}(N) \geq 1 - \alpha$ for $i \in S$. This problem could be termed the "lemon" problem since it represents a straightforward stochastic generalization of the deterministic maintenance case, where there is a probability of $1 - \alpha$ that maintenance is ineffectual (i.e., a lemon).

It is easily verified that under these conditions assumptions (1), (4), (6a) and (6b) are satisfied. Hence, the results of Theorems 1 and 2 are valid for any reward function satisfying assumptions (2), (3), and either (5a) or (5b). If both (5a) and (5b) hold, as will be the case under assumption (7), the optimal policy has a constant maintenance control limit structure.

A second example that satisfies assumptions (1), (4) and (6b) but not (6a) is the case where there is a single target state, which we denote by $i_0$, such that action $a$ taken in state $i$ causes the system to remain in state $i$ with probability $1 - \alpha$, or move to state $i_0$ with probability $\alpha$, where $\alpha$ is nonincreasing in $a$ for $a < N$. Thus, all maintenance actions attempt to improve the system to state $i_0$ and differ according to their probability of achieving this goal. If the "do nothing" action is taken then, similar to the previous example, the system deteriorates according to an upper triangular IFR transition matrix with $p_{ii}(N) \geq 1 - \alpha$, for $a \in X$, $i \in S$.

It is straightforward to verify that assumptions (1), (4) and (6b) are satisfied. Hence, for any reward functions satisfying assumptions (2), (3) and (5b) then by Theorem 2 the optimal policy has a nonincreasing control limit structure.

Finally, to illustrate the types of problems that can be addressed via the methods described here, we suggest a specific application. Consider a sterile manufacturing process in which a product is produced under extremely clean conditions. The state of the process can be taken to be the level of contamination of the manufacturing environment (e.g., bacteria count). As the level of contamination rises the percentage of defective items increases, resulting in lower revenues. Maintenance actions correspond to different sterilization procedures. If the cost of a sterilization procedure does not depend on the level of contamination, then separability assumption (7) is satisfied. If, in addition, the level of contamination after sterilization does not depend on the level of contamination before sterilization, which might be the case if opening the equipment for any maintenance action thoroughly contaminates it, then state independent maintenance assumption (8) also holds.

Conclusions and Further Work

We have considered a class of maintenance problems in which multiple maintenance actions are available and states are completely observable. Sufficient conditions for the optimal policy to have an easily implementable control limit structure were given. We have also specified additional conditions under which the optimal maintenance policy is monotonic or constant. These conditions were shown to be applicable to specific maintenance problems.

To further increase the usefulness of this type of research in practical maintenance problems, three areas of research need to be pursued. First of all, maintenance problems with multiple maintenance actions but incompletely observable states need attention. These problems are significantly more difficult than their completely observable counterparts and do not result in such strong structural conclusions (see Hopp and Wu [8]). Secondly, more complex multicomponent problems need to be studied from a maintenance perspective as opposed to the more common replacement approach (see Haurie and L’Ecuyer [7], L’Ecuyer and Haurie [11] for examples of multicomponent replacement models). Since most equipment subject to deterioration consists of multiple components, such research would be of significant practical importance. Finally, more explicit modeling of the decision.

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between maintenance and replacement is needed, particularly when the replacement technology is subject to technological improvement. In this commonly encountered situation, the decision-maker must choose between repairing the old system and replacing it with a new, technologically superior one. Very few analytical models exist for analyzing this type of decision.

REFERENCES


Appendix

Lemma 2: Under assumptions (1) and (2), \( f_i^t \) is nonincreasing in \( i \).

Proof: Let \( f_i^t \) represent the maximal value function in state \( i \) with \( k \) periods remaining in a finite horizon problem and define the finite horizon equivalent to DPE (1) as

\[
f_i^t = \max_{a \in X_i} \{ R(i,a) + \beta \sum_{j=1}^n p_{ij}(a)f_j^{t+1} \}
\]

\[
f_i^0 = 0
\]

for \( i \in S \). Clearly \( f_i^0 \) is nonincreasing in \( i \). Suppose that \( f_i^{t-1} \) is nonincreasing in \( i \). Then it follows from the fact that \( X_i \supset X_{i+1} \) and assumptions (1), (2) and Lemma 1, that

\[
f_i^t = \max_{a \in X_i} \{ R(i,a) + \beta \sum_{j=1}^n p_{ij}(a)f_j^{t-1} \} \geq \max_{a \in X_{i+1}} \{ R(i,a) + \beta \sum_{j=1}^n p_{ij}(a)f_j^{t-1} \} \]

\[
= f_{i+1}^t
\]

for all \( t = 1, \ldots, n-1 \). Therefore, \( f_i^t \) is nonincreasing in \( i \) for all \( k \), and because \( f_i^t = \lim_{k \to \infty} f_i^k \), induction completes the proof.

Lemma 3: If \( \bar{a} \leq a \) then the following conditions are equivalent:

a) \( \sum_{j=1}^n (p_{ij}(\bar{a}) - p_{ij}(a)) \) is nonincreasing (nondecreasing) in \( i \) for \( k = 1, \ldots, n \).

b) \( \sum_{j=1}^n (p_{ij}(\bar{a}) - p_{ij}(a))g(j) \) is nonincreasing (nondecreasing) in \( i \) for any bounded nonincreasing function \( g \).

Proof: We prove the nondecreasing case. The nondecreasing case is entirely analogous.

(b) \( \rightarrow \) (a): Assume condition (b) holds. Then, in particular, the characteristic function

\[
g_e(j) = \begin{cases} 
1, & j \leq k \\
0, & \text{otherwise}
\end{cases}
\]

is nonincreasing in \( j \). But we have

\[
\sum_{j=1}^n (p_{ij}(\bar{a}) - p_{ij}(a))g_e(j) = \sum_{j=1}^i (p_{ij}(\bar{a}) - p_{ij}(a))
\]

and hence, condition (a) holds.
(a) – (b): Assume condition (a) holds. Since \( g \) is bounded we can write \( g(j) = h(j) + c \), where \( h \) is a nonnegative function and \( c \) is a constant. Also, since \( g \) is a simple function, we can express \( h(j) \) as

\[
h(j) = \sum_{k=1}^{n} \lambda_k g_k(j)
\]

where \( \lambda_k \geq 0 \), \( k = 1, \ldots, n \) and \( g_k(j) \) is defined above (see Royden [15]). Then

\[
\sum_{j=1}^{n} (p_0(\bar{a}) - p_0(\tilde{a})) g(j) = \sum_{j=1}^{n} (p_0(\bar{a}) - p_0(\tilde{a})) h(j)
\]

\[
= \sum_{j=1}^{n} (p_0(\bar{a}) - p_0(\tilde{a})) \sum_{k=1}^{n} \lambda_k g_k(j)
\]

\[
= \sum_{k=1}^{n} \lambda_k \sum_{j=1}^{n} (p_0(\bar{a}) - p_0(\tilde{a})) g_k(j)
\]

\[
= \sum_{k=1}^{n} \lambda_k \sum_{j=1}^{n} (p_0(\bar{a}) - p_0(\tilde{a}))
\]

Since \( \lambda_k \geq 0 \), condition (b) holds. ■

**Theorem 1:** Under assumptions (1)-(4), the optimal policy has the property that for some \( i^* \), \( x^*_i = N \) for \( i < i^* \) and \( x^*_i < N \) for \( i \geq i^* \).

**Proof:** It is sufficient to show that if \( x^*_i = N \) then \( x^*_i = N \) for all \( i < k \). Fix any \( i < k \). Suppose \( x^*_i = N \) but \( x^*_i = a < N \). Then DPE (1) implies

\[
R(i,a) + \beta \sum_{j=1}^{n} p_0(\bar{a}) f_j > R(i,N) + \beta \sum_{j=1}^{n} p_0(N) f_j
\]

and

\[
R(k,N) + \beta \sum_{j=1}^{n} p_0(N) f_j \geq R(k,a) + \beta \sum_{j=1}^{n} p_0(\bar{a}) f_j
\]

which together imply

\[
\beta \sum_{j=1}^{n} (p_0(\bar{a}) - p_0(\tilde{a})) f_j + R(k,a) - R(k,N) < \]

\[
\beta \sum_{j=1}^{n} (p_0(\bar{a}) - p_0(\tilde{a})) f_j + R(i,a) - R(i,N)
\]

This inequality plus assumption (3) imply

\[
\sum_{j=1}^{n} (p_0(\bar{a}) - p_0(\tilde{a})) f_j < \sum_{j=1}^{n} (p_0(\bar{a}) - p_0(\tilde{a})) f_j.
\] (4)

However, Lemmas 2 and 3, along with assumption (4), imply

\[
\sum_{j=1}^{n} (p_0(\bar{a}) - p_0(\tilde{a})) f_j \geq \sum_{j=1}^{n} (p_0(\bar{a}) - p_0(\tilde{a})) f_j
\]

which contradicts (4). Hence, \( x^*_i = N \) for all \( i \leq k \). Defining

\[
i^* = \min \{ i : \max_{a \in X} [R(i,a) + \beta \sum_{j=1}^{n} p_0(a) f_j] > R(i,N) + \beta \sum_{j=1}^{n} p_0(N) f_j \}
\]

(5)

where \( i^* \) is taken to be \( n + 1 \) if no \( i \in S \) satisfies the inequality in (5), completes the proof. ■

**Theorem 2:**

a) Under assumptions (1)-(4), (5a) and (6a), \( x^*_i \) is nondecreasing in \( i \) for \( i \geq i^* \).

b) Under assumptions (1)-(4), (5b) and (6b), \( x^*_i \) is nonincreasing in \( i \) for \( i \geq i^* \).

**Proof of (a):** Suppose \( x^*_i = \bar{a} < N \) but \( x^*_i = \tilde{a} < \bar{a} \) for some \( k > i \). Then DPE (1) implies

\[
R(i,\bar{a}) + \beta \sum_{j=1}^{n} p_0(\bar{a}) f_j \geq R(i,\tilde{a}) + \beta \sum_{j=1}^{n} p_0(\tilde{a}) f_j
\]

and

\[
R(k,\tilde{a}) + \beta \sum_{j=1}^{n} p_0(\tilde{a}) f_j > R(k,\bar{a}) + \beta \sum_{j=1}^{n} p_0(\bar{a}) f_j
\]

which together imply

\[
\beta \sum_{j=1}^{n} (p_0(\bar{a}) - p_0(\tilde{a})) f_j + R(i,\tilde{a}) - R(i,\bar{a}) < \]

\[
\beta \sum_{j=1}^{n} (p_0(\tilde{a}) - p_0(\bar{a})) f_j + R(k,\bar{a}) - R(k,\tilde{a})
\]

Combined with assumption (5a), this implies

\[
\sum_{j=1}^{n} (p_0(\bar{a}) - p_0(\tilde{a})) f_j < \sum_{j=1}^{n} (p_0(\tilde{a}) - p_0(\bar{a})) f_j
\]

Lemmas 2 and 3 and assumption (6a) imply the reverse of this inequality, a contradiction. Hence, \( x^*_i \) is nondecreasing in \( i \).

**Proof of (b):** The proof is entirely analogous to that for (a). ■
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