Price Trends in a Dynamic Pricing Model with Heterogeneous Customers: A Martingale Perspective

Xiaowei Xu
Department of Supply Chain Management and Marketing Sciences, Rutgers, The State University of New Jersey, Newark, New Jersey 07102, xiaowei.xu@andromeda.rutgers.edu

Wallace J. Hopp
Stephen M. Ross School of Business, University of Michigan, Ann Arbor, Michigan 48109, whopp@umich.edu

This note describes probabilistic properties of optimal price sample paths in a dynamic pricing model with a finite horizon and limited stock. We assume that customer arrivals follow a nonhomogeneous Poisson process. We show that if customers' willingness-to-pay increases rapidly over time, then the optimal price process follows a submartingale, which implies an upward price trend. Alternatively, if customers' willingness-to-pay decreases rapidly over time, then the optimal price process follows a supermartingale, which implies a downward price trend.

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1. Introduction
The aim of this note is to characterize dynamic price processes in markets with an end date or a seasonal structure. A growing literature has addressed optimal dynamic pricing policies in such environments. For example, Gallego and van Ryzin (1994) developed a widely used framework, in which customer arrivals are modeled as a Poisson process. They studied the case with homogeneous demand (i.e., customers' willingness-to-pay is constant over time) and found that the optimal price is decreasing in the remaining inventory level at any given time (the inventory-monomonicity property) and also decreasing over time with a fixed inventory level (the time-monomonicity property). Zhao and Zheng (2000) allowed nonhomogeneous demand (e.g., customers' willingness-to-pay changing over time). They proved that the inventory-monomonicity property continues to hold and found a sufficient condition for the time-monomonicity property. The sufficient condition requires a decrease in customers' willingness-to-pay over time.

However, when an optimal pricing policy is implemented, both inventory levels and time change simultaneously. As illustrated in Gallego and van Ryzin (1994) and Bitran and Mondschein (1997), an optimal price sample path usually exhibits a zig-zag shape as inventory levels decrease over time. Because of this, there are no mononic results for the optimal price process at the sample-path level.

In this note, we study the optimal price process from a new perspective. Specifically, we offer conditions under which a monotonic price trend is generated by an optimal pricing policy in the type of environment considered by Gallego and van Ryzin (1994) and Zhao and Zheng (2000). However, the “trend” to which we refer is described in a probabilistic sense. That is, instead of focusing on the sample-path level, we consider the mononicity of the optimal price process in expectation. We show that if customers' willingness-to-pay increases rapidly over time, then the optimal price process follows a submartingale, which implies an upward price trend. Alternatively, if customers' willingness-to-pay decreases rapidly over time, then the optimal price process follows a supermartingale, which implies a downward price trend.

2. The Model
We assume that a firm has a stock level of \( n \) (a non-negative integer) items at time 0, which it sells over the interval \([0,T] \). Customer arrivals follow a nonhomogeneous Poisson process with intensity \( \lambda(s) \), where \( s \in [0,T] \).

We assume that customers, who arrive at time \( s \), have a utility function \( V(s,p) = U(s) - \alpha(s)p \) and purchase the product if their utility value \( V(s,p) \) is larger than zero, where \( U(s) \) is the reservation value of the product, \( \alpha(s) \) is the price sensitivity, and \( p \) is the product price. Notice that customers' willingness-to-pay can be dynamic over time as a result of two effects: (a) customer price sensitivity...
α(s) changes over time, and (b) the reservation value U(s) changes over time.

We assume that customers are heterogeneous in their reservation value of the product; that is, U(s) is a random variable with probability distribution Φ(υ, s). We let φ(υ, s) be the density function of Φ(υ, s) and define the failure rate of Φ(υ, s) as h(υ, s) = φ(υ, s)/Φ(υ, s), where Φ(υ, s) = 1 − Φ(υ, s). This implies that the total demand rate at time s given price p is λ(υ, s) = λ(υ)Φ(α(υ)p, s). We assume that λ(υ, s) is twice differentiable in (p, s) and let ℙ = {p(s) | s ∈ [0, T]} be a nonanticipatory pricing policy that controls the actual customer demand process M(s; ℙ). Notice that if α(υ) = 1 for s ∈ [0, T], then we have the demand process studied in Zhao and Zheng (2000).

**Condition 1.** For every s ∈ [0, T], λ(υ, p) is log-concave in p, or equivalently, h(υ, s) is increasing in υ ∈ [0, +∞).

Condition 1 means that U(s) has an increasing failure rate for every s ∈ [0, T] (see Lariviere 2006 and Ziya et al. 2004 for applications of failure rates in supply chain management and revenue management). By Condition 1, β(υ, p) = log(λ(υ, p)) is concave in p.

At time s, with remaining inventory level l, the firm maximizations expected revenue J(l, s; ℙ) = E[∫_0^T p(t) dM(t; ℙ)] subject to the inventory constraint ∫_0^l dM(t; ℙ) ≤ l, where J(l, T; ℙ) = 0 and J(0, s; ℙ) = 0 for l ≤ n and s ∈ [0, T]. Notice that the inventory constraint can be satisfied by setting price to infinity after a stockout (called the null price in Gallego and van Ryzin 1994). We denote an optimal pricing policy by ℙ^∗ ∈ arg max_{p ∈ ℙ} J(l, s; ℙ), where ℙ represents the set of all nonanticipatory pricing policies satisfying the inventory constraint. This allows us to express the optimal expected revenue in [s, T] given inventory level l as J^*(l, s) = J(l, s; ℙ^∗) = sup_{p ∈ ℙ} J(l, s; ℙ).

By the Principle of Optimality, we obtain the Hamilton-Jacobi-Bellman equation for J^*:

\[-J^*_s(l, s) + sup_{p(s)} \lambda(υ(p), s) \{p(s) − I^*(l, l, s)\},\]

where

\[J^*_s(l, s) = \frac{∂J^*(l, s)}{∂s}, \quad I^*(l, s) = J^*(l, s) − J^*(l − 1, s), \quad 1 ≤ l ≤ n,\]

and s ∈ [0, T] (see derivation details in Gallego and van Ryzin 1994 and Zhao and Zheng 2000). By Condition 1, there exists a unique optimal solution p^*(l, s) for the right side of the above equation. Hence, the optimal pricing policy is given by ℙ^∗ = {p^*(l, s) | 1 ≤ l ≤ n, s ∈ [0, T]}.

Although ℙ^∗ and J^* must be computed numerically for most cases, there exists an implicit structure between ℙ^∗ and J^*, which enables us to prove a martingale property for the optimal pricing process by applying Dynkin’s formula (Rolski et al. 1999). Dynkin’s formula has been used by others to obtain optimal expected revenue functions (see, e.g., Feng and Xiao 2000, Feng and Gallego 2000).

Let N(s) be a nonhomogeneous Markovian process, which models the inventory process driven by the optimal pricing policy ℙ^∗ (see details in Online Appendix I, which is available as part of the online version that can be found at http://or.pubs.informs.org/), and ℙ^N be the σ-algebra generated by {N(t), t ≤ s}. We define an optimal stopping time \( τ = \inf\{s ∈ [0, T] | N(s) = 0\} \), which represents the time when the inventory level falls to zero, and the optimal price process

\[ P(s) = \begin{cases} p^*(N(s), s) & \text{if } s < \tau \\ p^*(1, \tau−) & \text{if } s ≥ \tau. \end{cases} \]

Because the optimal pricing policy can take an arbitrarily defined value when a stockout occurs, we stop the price process P(s) immediately after stockouts occur and let the price process take the value it had just before a stockout occurs. In Online Appendix II, we study an alternative price process, in which we stop the price process when inventory level falls to one, and show that similar monotonicity results to those presented below also hold for this alternative price process under weaker conditions.

### 2.1. Upward Price Trends

We first establish that the optimal price process is a submartingale, and hence the price trend is upward under the two conditions stated below.

**Condition 2.** For every s ∈ [0, T], \( 1/h(υ, s) \) is convex in υ.

Condition 2 holds for uniform, logistic, and normal distributions and gamma and Weibull distributions with shape parameters no less than one. These distributions also have increasing failure rates, and hence satisfy Condition 1 (proofs of this and all other results are available in Online Appendix I).

We offer an economic interpretation of Condition 2 by adopting an idea used in Tyagi (1999). We define the marginal revenue function

\[ mr(p, s) = \frac{∂}{∂p} \left[ pλ(υ, p) \right] \]

\[ = p + \frac{λ(υ, p)}{∂p} \frac{∂λ(υ, p)}{∂p} = p − \frac{1}{α(υ)h(α(υ)p, s)}. \]

This implies that

\[ \frac{∂^2 [mr(p, s)]}{∂p^2} = −α(υ) \frac{∂^2}{∂υ^2} \left[ \frac{1}{h(υ, s)} \right]_{υ=α(υ)p}. \]

Hence, Condition 2 is equivalent to the condition that the marginal revenue function \( mr(p, s) \) is concave in p.
Let \( \tilde{A} = \sup_u \Phi(u, T)u \) and \( B = \sup_{s \in [0, T]} \lambda(s) \). By the Markov inequality, \( \tilde{A} \leq E[U(T)] \). Let

\[
\tilde{G} = \sup_{(u, s) \in [0, +\infty) \times [0, T]} u \Phi(u, s), \quad \text{where} \quad u_0 = \arg \max_u [\Phi(u, 0)u].
\]

We define \( \tilde{H}_1 = BG \) and \( \tilde{H}_2 = B\tilde{A} \).

**Condition 3.** Either (1) \( \sup_{s \in [0, T]} [\log(\alpha(s))] \leq -\tilde{H}_1 \), and \( h(u, s) \) is decreasing in \( s \) for every \( u \); or (2) \( \alpha(s) \) is decreasing in \( s \), \( h(u, s) \) is decreasing in \( s \) for every \( u \), and \( \inf_{(u, s) \in [0, +\infty) \times [0, T]} (\partial/\partial s) [1/h(u, s)] > \tilde{H}_2 \).

If \( h(u, s) = h(u) \) (i.e., customers’ reservation value of the product is independent of time), then Condition 3.1 implies that price sensitivity \( \alpha(s) \) is rapidly decreasing in \( s \). If \( \alpha(s) \) is constant over time, Condition 3.2 implies that \( U(s_1) \leq h(U(s_2)) \), where \( 0 \leq s_1 \leq s_2 \leq T \) and \( \leq_{th} \) is the hazard rate order. By Theorem 1.3.8 of Müller and Stoyan (2002), \( U(s_1) \leq_{st} U(s_2) \), where \( \leq_{st} \) is the usual stochastic order. Hence, Condition 3.2 implies that customers’ reservation value is rapidly increasing over time.

Notice that

\[
\frac{\partial^2}{\partial p \partial s} [\log(\lambda(p, s))] = -\frac{\partial}{\partial s} [h(\alpha(s)p, s)\alpha(s)] = -[h_u(\alpha(s)p, s)\alpha(p) + h(\alpha(s)p, s)]\alpha'(s) - h(\alpha(s)p, s)\alpha(s) > 0,
\]

where the last inequality follows from Conditions 1 and 3. Hence, \( \lambda(p, s) \) is log-supermodular in \( (p, s) \). This implies that \( \lambda(p_L, s)/\lambda(p_r, s) = \Phi(\alpha(s)p_L, s)/\Phi(\alpha(s)p_r, s) \) is increasing in \( s \), where \( p_r > p_L \). Zhao and Zheng (2000) made a similar log-submodularity assumption (Assumption 1 therein) on \( \lambda(p, s) \) to establish the time-monotoncity property of the optimal pricing policy. By their interpretation, \( \Phi(\alpha(s)p_L, s)/\Phi(\alpha(s)p_r, s) \) is the conditional probability that a customer is willing to pay a higher price \( (p_r) \) given that she would like to buy at a lower price \( (p_L) \). With this interpretation, Condition 3 implies that customers’ willingness-to-pay is rapidly increasing over time because Condition 3 requires that \( \lambda(p, s) \) is “sufficiently” log-supermodular in \( (p, s) \). Finally, we notice that a rapid increasing of willingness-to-pay is a consequence of two effects: (a) customer price sensitivity is rapidly decreasing over time, and (b) reservation value of the product is rapidly increasing over time.

**Theorem 1.** If Conditions 1–3 hold, then \( P(s) \) is a \( \mathbb{F}_s \)-submartingale. In particular, \( E[P(s)] \) is increasing in \( s \in [0, T] \).

Hence, Conditions 1–3 represent sufficient conditions for the optimal price path to be increasing in expectation.

**Example 1.** We assume that \( \Phi(u, s) \) is independent of time (i.e., \( \Phi(u, s) = \Phi(u) \)). \( \Phi(\cdot) \) has an increasing failure rate \( h(\cdot) \), and \( 1/h(\cdot) \) is convex in \( u \). Hence, Conditions 1 and 2 are satisfied. Let \( \alpha(s) = e^{\delta s} \) and \( \lambda(p, s) = \lambda(\delta) \Phi(\alpha(s)p) \). If \( K < -\tilde{H}_1 \), then Condition 3.1 holds.

Because \( \Phi(u, s) = \Phi(u), \tilde{A} = \sup_u [\Phi(u)u], \) and \( \tilde{G} = \sup_{u \in [0, +\infty)} u \Phi(u), \) where \( u_0 = \arg \max_u [\Phi(u)u] \). If \( \Phi(\cdot) \) is an exponential distribution with mean one, then \( \tilde{A} = e^{-1}, \tilde{G} = e^{-1}, \) and \( u_0 = 1 \). Hence, \( \tilde{H}_1 = e^{-1}B \).

**Example 2.** Let \( \alpha(s) = 1 \) for \( s \in [0, T] \). We assume that \( \Phi(u, s) \) is a uniform distribution on \([\bar{U}, \bar{U}(s)]\) for every \( s \), where \( U > 0 \) and \( \bar{U}(s) \) is increasing in \( s \). Because \( 1/h(u, s) = \bar{U}(s) - u \), if \( \inf_{s \in [0, T]} \bar{U}(s) > H_2 \), then Condition 3.2 is satisfied. To calculate \( \tilde{H}_2 \), we notice that \( \tilde{A} = \bar{U}(T)^2/(4(\bar{U}(T) - \bar{U})). \)

As pointed out by Bitran and Monschein (1997) and Zhao and Zheng (2000), travelers are willing to pay more for their fares as the departure time approaches. By Theorem 1, we would expect an optimal pricing policy to exhibit an upward trend during the season if customers’ willingness-to-pay increases rapidly over time.

### 2.2. Downward Price Trends

In this section, we identify conditions that guarantee a downward price trend. We show that the conditions require a rapid drop of customers’ willingness-to-pay over time.

Let \( A = \sup_u [\Phi(u, 0)u] \). By the Markov inequality, \( A \leq E[U(0)] \). Let

\[
F = \inf_{s \in [0, T]} \frac{\partial}{\partial u} \left[ \frac{1}{h(u_T, s)} \right]
\]

\[
G = \sup_{(u, s) \in [0, +\infty) \times [0, T]} \Phi(u, s),
\]

where \( u_T = \arg \max_u [\Phi(u, T)u] \). Notice that \( F \leq 0 \) under Condition 1. We define \( H_1 = AB^2G(1 - F)T \) and \( H_2 = AB^2(1 - F)T \).

**Condition 4.** Either (1) \( \inf_{s \in [0, T]} [\log(\alpha(s))]' > H_1 \), and \( h(u, s) \) is increasing in \( s \) for every \( u \), or (2) \( \alpha(s) \) is increasing in \( s \), \( h(u, s) \) is increasing in \( s \) for every \( u \) and \( \sup_{u \in [0, +\infty) \times [0, T]} (\partial/\partial s) [1/h(u, s)] < -H_2 \).

Because

\[
\frac{\partial^2}{\partial p \partial s} [\log(\lambda(p, s))] = -[h_u(\alpha(s)p, s)\alpha(p) + h(\alpha(s)p, s)]\alpha'(s) - h(\alpha(s)p, s)\alpha(s),
\]

Condition 4 implies that \( \lambda(p, s) \) is “sufficiently” log-submodular in \( (p, s) \), that is, customers’ willingness-to-pay declines rapidly during the season. Condition 4.1 represents a sharp increase in price sensitivity and Condition 4.2 a substantial decrease in customers’ reservation value over time.
THEOREM 2. If Conditions 1, 2, and 4 hold, then $P(s)$ is a $\mathcal{F}_s^N$-supermartingale. In particular, $E[P(s)]$ is decreasing in $s \in [0, T]$.

Hence, Conditions 1, 2, and 4 represent sufficient conditions for the optimal price path to be decreasing in expectation.

**Example 3.** We assume that $\Phi(u, s) = \Phi(u)$, $\Phi(u)$ has an increasing failure rate $b(u)$, and $1/b(u)$ is convex in $u$. Let $\alpha(s) = e^{ks}$ and $\lambda(p, s) = \lambda(s)\Phi(\alpha(s)p)$. If $K > H_1$, then Condition 4.1 holds.

Because

$$\Phi(u, s) = \Phi(u), \quad A = \sup_u [\Phi(u)u], \quad F = \frac{\partial}{\partial u} [1/h(u_T)],$$

and

$$G = \sup_{u \in [0, \infty)} \Phi(u),$$

where $u_T = \arg \max_u [\Phi(u)u]$. If $\Phi(u)$ is an exponential distribution with mean one, then $A = e^{-1}$, $F = 0$, $G = e^{-1}$, and $g_T = 1$. Hence, $H_1 = e^{-2}B^2T$.

**Figure 1.** The mean of the optimal price process $P(s)$.

**Example 4.** Let $\alpha(s) = 1$ for $s \in [0, T]$. We assume that $\Phi(u, s)$ is a uniform distribution on $[\underline{U}, \bar{U}(s)]$ for every $s$, where $\underline{U} > 0$ and $\bar{U}(s)$ is decreasing in $s$. Because $1/h(u, s) = \bar{U}(s) - u$, if $\sup_{s \in [0, T]} \bar{U}(s) < -H_2$, then Condition 4.2 is satisfied. To calculate $H_2$, we notice that $A = \bar{U}(0)^2/(4\bar{U}(0) - \underline{U})$ and $F = -1$.

As pointed out by Bitran and Mondschein (1997) and Zhao and Zheng (2000), most early adopters are willing to pay more for fashion goods. Theorem 2 implies that if this decrease in willingness-to-pay is sufficiently large, the optimal price sample path will be decreasing in an expectation sense.

3. A Numerical Example

We revisit the numerical example in the bottom of Figure 1 in Gallego and van Ryzin (1994, p. 1005). Let $T = 1$, $n = 25$, and $\lambda(p, s) = 100e^{-4|\delta|}$, where $\alpha(s) = e^{ks}$ and $K \in \{-4, 0, 4\}$. We discretize time interval $[0, T]$ into $[0, \delta, 2\delta, \ldots, L\delta]$, where $L\delta = T$, and simulate the actual customer demand on $(j\delta, (j + 1)\delta)$ by a Bernoulli random variable with mean $\lambda(p^*(\cdot, j\delta), j\delta)$, where $j = 0, \ldots, L - 1$ (see

Note. (a) $K = -4$; (b) $K = 0$; (c) $K = 4$. 

Figure 1 shows the mean of the optimal price process \( P(s) \). Figure 1(a) shows an increase in optimal expected prices when the price sensitivity function decreases rapidly, which is consistent with Theorem 1. In contrast, Figure 1(c) shows a decrease in optimal expected prices when the price sensitivity function increases rapidly, which is consistent with Theorem 2. Notice that Conditions 3 and 4 are sufficient conditions for a monotonic trend of the optimal price process in expectation, but are not necessary conditions (e.g., \(|K| = 4\) does not satisfy Conditions 3 and 4 demonstrated in Examples 1 and 3). Finally, Figure 1(b) shows an initial increase in optimal expected prices, but a sharp decrease at the end of the period. Hence, the price trend is not monotonic when the price sensitivity function changes slowly over time.

4. Conclusion

The probabilistic characterization of optimal price sample paths given in this note offers an explanation of how price trends are affected by customers’ willingness-to-pay, which in turn depends on customers’ price sensitivity and reservation value of the product. A sufficiently rapid rise (decline) in willingness-to-pay implies an upward (downward) price trend.

5. Electronic Companions

An electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/. Online Appendix I includes technical details and all proofs. Online Appendix II presents the study of an alternative price process.

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References


