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Product Line Selection and Pricing with Modularity in Design

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This paper addresses the strategic impact of modular design on the optimal length and price of a differentiated product line. We represent consumer demand with a Bayesian logit model. We also break operations costs into product design and production components. Our analysis shows that reducing product development costs via modular design always makes it attractive to offer greater product variety. However, reducing production costs can sometimes motivate a *reduction* in variety for a risk-averse producer in a multiple-segment market. We also characterize the impacts of degree of modularity and production cost on price markup and market share. Finally, we show that the optimal product line length is monotonic in risk attitude and the monotonic weak majorization, partial order on product assortment.

Key words: majorization; modular design; multinomial logit; product differentiation; product variety
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1. Introduction and Literature Review

Determining the range of products to offer is one of the most critical strategic decisions faced by a firm. To effectively match diversified customer tastes, a firm is often tempted to offer a range of products. However, product proliferation generally increases product development and production costs. Hence, the conflicting impact of product variety on revenue and cost indicates that firms should focus on profit when making product line decisions, as suggested by Quelch and Kenny (1994). In recent decades, mass customization methods have evolved in an attempt to alleviate this conflict (Pine 1993). By mass customizing products, a firm can keep costs down and still extend its product line to generate sales. One of the most efficient tools of mass customization is modular design (Baldwin and Clark 1997). For instance, 85% of Sony's 250 new models launched during the 1980s involved only minor feature changes on cosmetic redesigns and shared many components, such as "superflat" motors and "chewing gum" batteries (Sanderson and Uzumeri 1995). See Table 1 of Sanchez (1999) for more applications of modular design.

It is widely believed that product variety increases a firm's market power, which further increases a firm's profit margin and market share. Several empirical studies support this belief. For instance, Kekre and Srinivasan (1990) study the relationships among product line length, market share, and price by exploring Profit Impact of Marketing Strategies (PIMS), a self-reported survey from more than 200 companies' data. Bayus and Putsis (1999) considered these same decision variables as jointly endogenous factors in their analysis of the computer industry over the period 1981–1992. Both indicated positive relationships among product line length, market share, and profit margin. Fisher et al. (1999) developed a specific quantitative model of automotive braking systems component sharing and empirically tested cost-saving strategies through modularity. For more empirical studies, see Labro (2004).

In this paper, we study a product line selection and pricing problem. We assume flexible production and distribution systems with no economies of scale, but a costly product design process with economies of scope, which can be improved by modular design. We focus on identifying strategic insights into the impacts of degree of modularity and production cost on product line length, price markup, and market share. The economics literature has studied the effect of product variety from a market perspective by simply assuming a linear product development cost function (see Lancaster 1990 for a review). In contrast, the operations management literature generally views modular design as a cost-saving tool (see, e.g., Ulrich and Eppinger 2004). Our model brings the market competition and cost-reduction perspectives together and studies their interactions. For the most part, our theoretical work supports the general belief that modularity facilitates product variety and some empirical observations that a broader product line results in higher market share and price (Kekre and Srinivasan 1990, Hypotheses 1 and 2).

The trade-off between cost and revenue in product variety problems has also been studied in the product-positioning and development literature. A product-positioning problem addresses decisions of how many products to offer, what features to include, and what pricing strategy to pursue (for literature reviews see Green and Krieger 1989, Kaul and Rao 1995). The problem typically involves consumer choice constraints; that is, consumers pick the product that maximizes their utility function. Product development cost is sometimes considered through a fixed cost per product (see, e.g., Dobson and Kalish 1993). Our paper differs from the product-positioning literature with regard to both methodology and research objective. Product-positioning problems are often solved by mixed integer programming (MIP) techniques and heuristic methods, which are robust to variations in practice but ill suited for summarizing relationships among key factors and gaining managerial insights. In contrast, we make use of a simple and stylized model that only involves minor optimization issues and is thereby capable of capturing managerial insights to important factors.

In the product development literature, Desai et al. (2001) provide a conceptual design configuration model of product quality via sharing components in a market with two segments (high and low), study the balance between cost and revenue, and identify conditions when commonality should be used in two differentiated products. Kim and Chhajed (2002) consider a product line design model in which a monopolist offers two multiattribute products to serve a market with two customer segments. Both papers involve only two products and are therefore limited with respect to what they can say about the importance of modularity. Our paper allows an arbitrary number of products (a broad product line), which enables us to directly examine the power of modularity. See Ho and Tang (1998), Krishnan and Ulrich (2001), and Ramdas (2003) for reviews of the product variety and development literature.

Throughout this paper, we assume that demand follows a Bayesian logit model. A logit model is a good representation of demand for horizontally differentiated products. It is also a tractable way to model a product line with a large number of variants. Some product design literature has made use of a logit profit function (i.e., demand is generated by a logit model). Hanson and Martin (1996) develop a path-following procedure (homotopy algorithm) for optimizing a logit profit function. Chen and Hausman (2000) present a similar model for maximizing logit profit, but under their modeling assumptions integrality constraints can be relaxed, which leads to an efficient optimal algorithm. Neither paper included product development cost.

Aydin and Ryan (2000) is the work most closely related to our paper, but these authors did not explicitly model product development cost or the impact of modular design on product line length. They did, however, show that variant price markups are equal under logit demand, a result also shown for the nested logit model by Anderson and de Palma (1992). We extend this property to the case with a random brand effect.

In this paper, we introduce a new partial order for product assortment called monotonic weak majorization, which is a refinement of the majorization order introduced by van Ryzin and Mahajan (1999). van Ryzin and Mahajan studied the trade-off between inventory costs and variety benefits in retail assortments, using a logit model to describe demand and a newsvendor model to represent the retailer's inventory cost. They proved that profit increases along the majorization order. In our model, we find that majorization guarantees profit monotonicity, but is not enough for monotonicity in product variety, which only holds under monotonic weak majorization.

In this paper, we assume that a producer is risk sensitive with an exponential utility function. Although modern financial theory implies that firms should maximize expected net present value, there are two reasons firms may behave in a risk-sensitive manner. First, factors such as nondiversified owners, imperfect information in the capital market, and costly financial distress may present unhedged risks. For instance, Greenwald and Stiglitz (1990) conclude that firms are risk averse as a result of imperfect information in the capital market. In our problem, because product development often involves huge capital investment (relative to firm size), it is important to avoid "risking bankruptcy" (Ramirez 1989). Second, the compensation scheme of a risk-averse manager is often linked to firm performance. Walls and Dyer (1996) applied an exponential utility function in the petroleum exploration industry and discussed the appropriateness of assuming a risk-averse firm.

The remainder of the paper is organized as follows. Section 2 presents the model. All comparative statistics are provided in §3. Finally, we discuss our conclusions and future research topics in §4. All omitted proofs are given in the appendix.

2. The Model

We consider here a producer with capacity to offer a finite number of variants that compete with an outside good, which can also be interpreted as the choice to opt out. In such environments the producer must choose its product line and prices. To frame these decisions in a model, we make the following assumptions on the supply side.

Assumption 1. *The producer has a menu of potential variants, indexed by $M = \{q_j\}_{j=1}^{n_0}$, where n_0 is the maximum number of variants that the producer is able to offer and q_j is the quality index of variant j.*

The potential number of variants, n_0, is countable but may be very large. For instance, an infinite number of colors of paint can be made by mixing several basic colors and dozens of pigments; countability is ensured by the resolution of the process. We denote an offered menu, which is a subset of M, by m. The set of all possible menus is denoted by $\mathscr{A}(M)$.

Assumption 2. *The product development cost is an increasing function of the number of variants in an offered menu.*

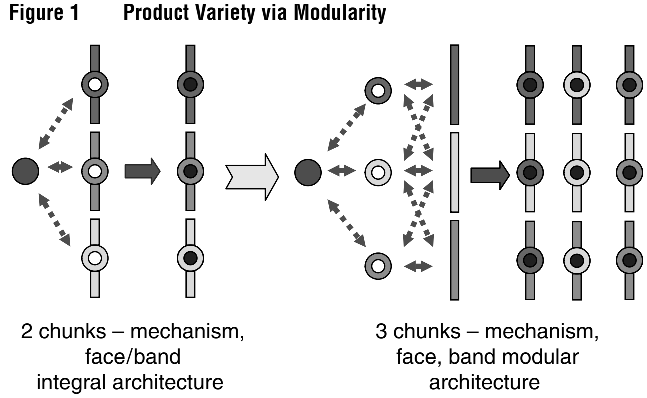
We denote the product development cost of offering a menu m by $FC(m) = c_f \times g(|m|)$, where $|m|$ is the number of variants in menu m and g is increasing. Note that, if g is concave, the product development cost exhibits economies of scope.

Product development cost is heavily determined by product complexity, which is a measure of the number of parts, interfaces, and so on (see, e.g., Meyer and Lehnerd 1997, p. 97). Modular design is an effective way to limit the need for new components, reduce product complexity, save on product design cost, and hence facilitate mass customization and product variety (see discussion of the Black & Decker case in Pine 1993, p. 199).

Figure 1 illustrates the basic mechanics of modular design using the case of Swatch, which offers hundreds of watches by combining standardized hands, faces, and wristbands (Ulrich and Eppinger 2004, p. 169). For example, using an *integral architecture*, three bands and three faces result in only three variants. However, using *modular architecture*, the three faces and three bands can be combined into nine variants. The key to modular architecture is to divide a product into modules, or "chunks," which can be varied independently to produce variety in the end product. Ulrich (1995) discussed a similar combinatorial example of developing trailers.

To characterize product design costs in a modular environment, we develop a cost model similar to the optimization model in Labro (2004). We let h_d represent the unit design cost for the dth chunk,

Figure 1 **Product Variety via Modularity**



$d=1,\ldots,K$. If, for the sake of illustration, consumers value variety in all modules equally, we can formulate the problem of finding the minimum cost to develop n variants as $\min_{T_d}\sum_{d=1}^{K}T_d h_d$, s.t. $\prod_{d=1}^{K}T_d=n$, where T_d is the number of options for the dth chunk. By the KKT condition, $h_d T_d=\lambda n$, where $\lambda=n^{(1-K)/K}[\prod_{d=1}^{K}h_d]^{1/K}$, which implies that we will offer more variety in chunks where development costs are low. The optimal number of options for the dth chunk is

$$T_d^*=\left[\prod_{j=1}^{K}h_j\right]^{1/K}\frac{n^{1/K}}{h_d}$$

and the minimal design cost is $c_f n^{1/K}$, where $c_f=K[\prod_{d=1}^{K}h_d]^{1/K}$. If $h_d\sim 1/K$ (i.e., chunks are equally costly to develop), then each chunk should offer an equal number of options and c_f does not depend on K.

This argument shows that if c_f is independent of K, which is actually true when design costs of different chunks are equal, then the product design cost of a menu of width $|m|$ is $FC(m)=c_f|m|^{1/K}$. In this expression, c_f characterizes the magnitude of the development costs and K characterizes efficiencies due to the degree of modularity. By the definition of the degree of commonality in Collier (1982), a menu with n variants has degree of commonality $n^{(K-1)/K}$, which is increasing in K for fixed n. Hence, the degree of modularity K also captures the degree of commonality in our model. Of course, in real systems, some combinations of chunk options may be infeasible (e.g., due to color mismatch), chunk development costs may vary, and consumers may value variety in some chunks more than others. However adjustment of c_f and K can provide approximations or bounds for the development cost in these cases. Hence, for purposes of studying the effect of modularity on product variety, we will use this functional form to represent the fixed development costs. The concave shape of $FC(m)$ matches qualitative behavior cited earlier for the Sony (Sanderson and Uzumeri 1995) and Black & Decker (Pine 1993) cases, in which modularity reduced the unit development costs of additional variants. Moreover, we will see that product variety is much more heavily determined by the degree of modularity (K) than by the cost magnitude (c_f). Hence, this simplification does not strongly affect our overall conclusions.

Assumption 3. *The production system is efficient and flexible* (*i.e., there are no economies of scale*), *so production cost can be approximated as a linear function of production volume.*

Under this assumption, the variable operating cost of producing x_j units of the jth variant can be written as $VC(x_j)=c_j\times x_j$, where c_j is the unit variable cost of variant j.

Assumption 4. *The producer has exponential utility* $U_\gamma(x)=(1-e^{-\gamma x})/\gamma$ *with* $\gamma\neq 0$ *or* $U(x)=x$ *if* $\gamma=0$.

The parameter γ measures risk attitude; when $\gamma=0$ (>0, <0), the producer is risk neutral (risk averse, risk seeking).

We further make two assumptions on the demand side.

Assumption 5. *Demand for offered variants and the outside good follows a logit model with random brand effects.*

Specifically, if menu $m=(q_1,q_2,\ldots,q_n)$ is offered with prices $\mathbf{p}=(p_1,p_2,\ldots,p_n)$, the probabilities of buying variant $j=1,\ldots,n$, or the outside good are

$$r_j=E^{\delta_\mu}\left\{\frac{e^{(q_j-p_j)/\mu}}{e^{\delta_\mu}+\sum_{l=1}^{n}e^{(q_l-p_l)/\mu}}\right\}\quad\text{and}$$

$$r_0=E^{\delta_\mu}\left\{\frac{e^{\delta_\mu}}{e^{\delta_\mu}+\sum_{l=1}^{n}e^{(q_l-p_l)/\mu}}\right\},\quad\text{where }\delta_\mu=\frac{b_0-b_1}{\mu}$$

and the random variables b_0 and b_1 are measures of the *brand effect* of the outside good and the producer's product line. Hence, δ_μ measures the brand advantage of the product line. Gönül and Srinivasan (1993), who give a general discussion of Bayesian logit models and random brand effects, describe the randomness of b_0 and b_1 as capturing "variations on the intrinsic brand utility across households" (p. 216). We further assume that $E^{\delta_\mu}[e^{-\delta_\mu}]<+\infty$, which is satisfied by the normal distribution and any distribution with bounded support. The parameter μ represents the unobservable heterogeneity of consumer taste; larger values of μ indicate smaller effects of price and brand (see Anderson et al. 1992 for more modeling details of consumer heterogeneity).

Assumption 6. *Demand can be backlogged without penalty.*

Assumptions 3 and 6 rule out economies of scale in the production and distribution processes, so we focus attention on the economies of scope generated by the product design cost.

Let N be the potential market size (i.e., number of consumers), which is fixed. By the above assumptions, the expected total profit is $E\{\Pi(\mathbf{p}, m)\} = N\sum_{j=1}^{n}(p_j - c_j)r_j - c_f g(|m|)$. For convenience, we define price markups $\bar{p}_j = (p_j - c_j)/\mu$, quality markups $\bar{q}_j = (q_j - c_j)/\mu$, $j = 1, \ldots, n$ and $\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n)$. Then, expected profit can be written as $E\{\Pi(\bar{\mathbf{p}}, m)\} = N\mu\sum_{j=1}^{n}\bar{p}_j r_j - c_f g(|m|)$, where

$$r_j = E^{\delta_\mu}\left\{\frac{e^{\bar{q}_j - \bar{p}_j}}{e^{\delta_\mu} + \sum_{j=1}^{n} e^{\bar{q}_j - \bar{p}_j}}\right\}.$$

Similarly, expected utility is given by

$$E\{e^{-\gamma\Pi}\} = \left(r_0 + \sum_{j=1}^{n} r_j e^{-\gamma(p_j - c_j)}\right)^N e^{\gamma c_f g(|m|)}$$
$$= \exp\left\{N\ln\left(r_0 + \sum_{j=1}^{n} r_j e^{-\gamma\mu\bar{p}_j}\right) + \gamma c_f g(|m|)\right\}.$$

Given the offered menu m, the problem is to choose the vector of price markups $\bar{\mathbf{p}}$ that maximizes expected total profit $E\{\Pi(\bar{\mathbf{p}}, m)\}$ when the producer is risk neutral or expected utility $E\{U_\gamma(\Pi(\bar{\mathbf{p}}, m))\}$ when the producer is risk sensitive. It is easy to check that if $\gamma\mu \leq -1$, setting price markups as large as possible always makes the producer better off. Because this does not yield a well-defined model, we assume that $\gamma\mu > -1$ from now on.

Proposition 1. *Given menu m, the optimal price markups of the n variants are positive and equal.*

Similar results were obtained by Anderson and de Palma (1992) for a nested-logit model and Aydin and Ryan (2000) for a logit model. Proposition 1 shows that their equal pricing results extend to the Bayesian logit model with risk-sensitive producers. The underlying reason for this result is that logit-type models imply that cross-price elasticities are equal for all pairs of variants (see the proof of Proposition 1). Equal cross-price elasticities often hold for horizontally differentiated products, such as lipstick (differentiated by color), greeting cards (differentiated by design), and ice cream (differentiated by flavor), but not for vertically differentiated products, such as the Toyota Corolla, Camry, and Lexus. For vertical differentiation models, only cross-price elasticities between adjacent products are nonzero (e.g., demand for Lexus is sensitive to Camry price but not to Corolla price).

Proposition 1 allows us to simplify the expected total profit and utility functions to

$$E\{\Pi(\bar{p}, m)\} = N\mu E^{\delta_\mu}\left\{\frac{\bar{p}e^{-\bar{p}}\sum_{j=1}^{n} e^{\bar{q}_j}}{e^{\delta_\mu} + e^{-\bar{p}}\sum_{j=1}^{n} e^{\bar{q}_j}}\right\} - c_f g(|m|) \quad \text{and}$$

$$E\{e^{-\gamma\Pi(\bar{p}, m)}\} = \exp\left\{N\ln\left(E^{\delta_\mu}\left\{\frac{e^{-(\gamma\mu+1)\bar{p}}\sum_{j=1}^{n} e^{\bar{q}_j} + e^{\delta_\mu}}{e^{\delta_\mu} + e^{-\bar{p}}\sum_{j=1}^{n} e^{\bar{q}_j}}\right\}\right) + \gamma c_f g(|m|)\right\}.$$

Recall that the menu of potential variants is $M = \{q_j\}_{j=1}^{n_0}$. Define $\{\bar{q}_{(j)}\}_{j=1}^{n_0}$ to be a decreasing rearrangement of $\{\bar{q}_j\}_{j=1}^{n_0}$, i.e., $\bar{q}_{(j)} \geq \bar{q}_{(i)}$ for all $j < i$, and let variant value $\bar{v}_{(j)} = e^{\bar{q}_{(j)}}$.

Proposition 2. *For any fixed price markup and product line length n, the optimal menu is composed of the variants with the largest n quality markups.*

We denote this menu by m^*. Proposition 2 means that it is always better to offer variants with high quality and low cost (high-quality markup) first, which implies $m^* = \{q_{(j)}\}_{j=1}^{n}$. This result was also obtained by Aydin and Ryan (2000, Corollary 6.2) for a logit model, and by van Ryzin and Mahajan (1999, Theorem 1) for ranking consumer preferences but without considering production costs. Both assumed risk-neutral agents. Proposition 2 shows that their results extend to the Bayesian logit model with risk-sensitive producers.

Note that m^* depends on n but not on $\bar{p}$, so we express the optimal menu of size n as $m^*(n)$. Let product line value $S(n) = \sum_{j=1}^{n}\bar{v}_{(j)}$. If the producer offers horizontally differentiated variants and $\bar{v}_{(j)}$ is constant, then $S(n)$ is linear in n. We now consider perturbations of the random brand effects $b_0' = b_0 + \Delta_{b,0}$ and $b_1' = b_1 + \Delta_{b,1}$, unit variable costs $c_j' = c_j + \Delta_c$, and quality indices $q_j' = q_j + \Delta_q$. Then, we can write the resulting brand advantage, quality markups, variant values, and product line value, respectively, as

$$\delta_\mu' = \delta_\mu + \frac{\Delta_{b,0} - \Delta_{b,1}}{\mu}, \quad \bar{q}_j' = \bar{q}_j + \frac{\Delta_q - \Delta_c}{\mu},$$
$$\bar{v}_{(j)}' = e^{(\Delta_q - \Delta_c)/\mu}\bar{v}_{(j)}, \quad \text{and} \quad S'(n) = e^{(\Delta_q - \Delta_c)/\mu}S(n).$$

By a slight abuse of notation, we denote the modified profit by

$$E\{\Pi'(\bar p, n)\} = N\mu E^{\delta'_\mu}\left\{\frac{S'(n)\bar p e^{-\bar p}}{e^{\delta'_\mu}+S'(n)e^{-\bar p}}\right\} - c_f g(n)$$
$$= N\mu E^{\delta_\mu}\left\{\frac{e^{\alpha}S(n)\bar p e^{-\bar p}}{e^{\delta_\mu}+e^{\alpha}S(n)e^{-\bar p}}\right\} - c_f g(n),$$
$$\text{where}\quad \alpha = \frac{\Delta_q - \Delta_c + \Delta_{b,1} - \Delta_{b,0}}{\mu}.$$

Similarly,

$$E\{e^{-\gamma\Pi'(\bar p,n)}\} = \exp\left\{N\ln\left(E^{\delta_\mu}\left\{\frac{e^{\alpha}S(n)e^{-(\gamma\mu+1)\bar p}+e^{\delta_\mu}}{e^{\delta_\mu}+e^{\alpha}S(n)e^{-\bar p}}\right\}\right) + \gamma c_f g(n)\right\}.$$

Hence, α captures the effects of perturbing brand effects, production costs, and quality indices and implies an equivalence among them. Although α measures changes in three factors, we call it the *production cost reduction* (to emphasize the production cost effect on α). Therefore, $e^{\alpha}S(n)$ represents the *market power* of the producer due to a change in brand equity, product line value, and production and design technologies.

We can further characterize the model by use of some additional standardized parameters: the *normalized magnitude of design cost* $\beta = c_f/(N\mu)$, the *normalized risk coefficient* $\Gamma = \gamma\mu$, and the *normalized market share* $s = (1/\mu)\sum_{j=1}^{n} r_j$. If $\mu = 1$, $\bar p$ is exactly the price markup and s is the market share. Finally, Γ captures both the risk attitude of the producer and the uncertainty of the market in the producer's perception.

Because we always offer the variants with the largest quality markups, the problem is to determine the optimal price markup $\bar p^*$ and the optimal product line length (number of variants) n^*. It is easy to see that maximization of expected total profit and utility is equivalent to maximizing

$$\pi(\bar p, n) = \begin{cases} E^{\delta_\mu}\left\{\dfrac{e^{\alpha}S(n)\bar p e^{-\bar p}}{e^{\delta_\mu}+e^{\alpha}S(n)e^{-\bar p}}\right\} - \beta g(n), & \Gamma = 0, \\ -\dfrac{1}{\Gamma}\ln\left(E^{\delta_\mu}\left\{\dfrac{e^{\alpha}S(n)e^{-(\Gamma+1)\bar p}+e^{\delta_\mu}}{e^{\alpha}S(n)e^{-\bar p}+e^{\delta_\mu}}\right\}\right) - \beta g(n), & \Gamma \neq 0. \end{cases}$$

We call π the transformed expected profit (utility) function. When $\Gamma = 0$, π is the expected profit per capita. It is easy to see that the maximum of $\pi(\bar p, n)$ with respect to $\bar p$ always exists. We allow n to take nonnegative integer values, although many results also hold for nonnegative real values (except for the results in §3.2 and §3.5). Note that because the maximum product line length is n_0, an optimal solution always exists.

3. Comparative Statics

Broadening a product line will result in an increase in product development cost but will allow firms to satisfy diversified consumer tastes better, charge a higher price premium, and gain a larger market share. The cost and revenue trade-off makes product line and price decisions sensitive to firms' technological factors (design and production costs). These exogenous factors, plus endogenous decision variables (product variety and prices), are vital to the success of any product line extension and hence have been widely studied empirically (e.g., Bayus and Putsis 1999, Kekre and Srinivasan 1990).

In this section, we seek insights into these trade-offs by analytically examining the qualitative impact of the key parameters, such as the magnitude of design cost (β), degree of modularity (K), and production cost reduction (α), on the optimal product line length n^*, price markup $\bar p^*$, and market share s^*. We also study the effects of risk coefficient (Γ) and majorization (which will be introduced later) on product variety, price markup, and market share.

Throughout this section, we refer to the case with $\delta_\mu = 0$ as a *single-segment market*; otherwise, we call it a *multiple-segment market*. We will explicitly identify this market condition only where a claim holds for a single-segment market but is not valid for a multiple-segment market.

3.1. Magnitude of Design Cost β

Proposition 3. *The optimal product line length is decreasing in the magnitude of design cost.*

Proposition 4. *The optimal market share is decreasing in the magnitude of design cost.*

Propositions 3 and 4 imply that the firm should offer more variants and will achieve a larger market share if the product development cost c_f is reduced. In addition, Proposition 4 and Lemma 2 show that market share increases in the product line length n^*, which is consistent with the findings in Kekre and Srinivasan (1990). Although this may seem intuitive, it is not obvious because a producer with an expanded product line could raise prices and thereby lower market share to either exploit additional consumer surplus or recoup product development cost. Indeed, Bayus and Putsis (1999) found that the impact of product proliferation on net market share is negative in the personal computer industry, although still positive when the price effect is excluded.

In a multiple-segment market, the optimal price markup is often not monotonic, as we show below in Example 1 (see more examples in Xu 2005), but the monotonicity of the optimal price markup does hold in a single-segment market.

Proposition 5. *In a single-segment market, the optimal price markup is decreasing in the magnitude of design cost.*

Proposition 5 implies that low product design cost stimulates a broad product line, which implies high price markups in a single-segment market. This is consistent with Hypothesis 2 in Kekre and Srinivasan (1990) and Hypothesis 3 in Bayus and Putsis (1999).

3.2. Degree of Modularity K

We now examine the impact of modular design on product line length. For this purpose, we use the specific form of the development cost function, $FC(m) = c_f g(n)$, where $n = |m|$ is the number of variants and $g(n) = n^{1/K}$. Hence, increased modularity is captured by a reduction in development cost.

Proposition 6. *The optimal product line length is increasing in the degree of modularity.*

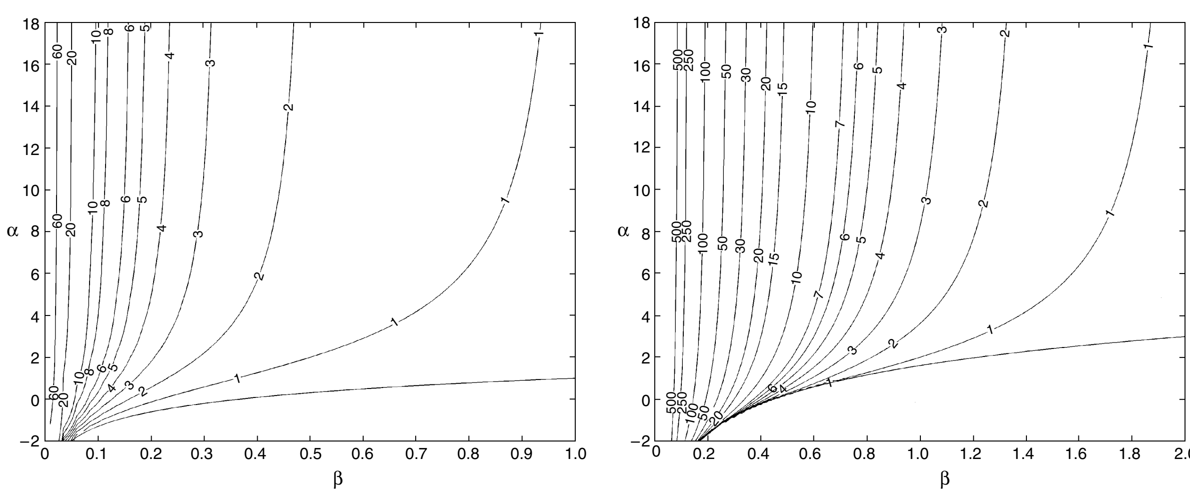
Proposition 7. *The optimal market share is increasing in the degree of modularity.*

Proposition 8. *In a single-segment market, the optimal price markup is increasing in the degree of modularity.*

We now consider two specific cases of the product development cost, $g(n) = n$ (no modularity) and $g(n) = \sqrt{n}$ (modular design with two chunks), for the case of horizontally differentiated products (i.e., with $S(n) = n$). We illustrate the effects of varying α (the production cost reduction) and β (the design cost magnitude) in Figure 2.

Figure 2 shows that the optimal line length (n^*) is very sensitive to the ratio of product development cost to market size (β). If the product development cost is very low, then it is inexpensive to introduce variants and hence it becomes optimal to offer many variants (Proposition 3). By comparing Figures 2(a) and 2(b), we see that using modular design and improving the degree of modularity greatly facilitates product variety (Proposition 6) and even has a larger scale effect on product variety than does the magnitude of design cost.

Figure 2 **The Contour of n^* when (a) $g(n) = n$, (b) $g(n) = \sqrt{n}$**



(a) (b)

3.3. Production Cost Reduction α

PROPOSITION 9. (1) *For a risk-neutral or risk-seeking producer* ($-1 < \Gamma \leq 0$), *the optimal product line length is increasing in the production cost reduction*; (2) *for a risk-averse producer* ($\Gamma > 0$) *in a single-segment market*, *the optimal product line length is increasing in the production cost reduction*.

As shown in Figure 2, the optimal product line length is relatively insensitive to α ($\gg 0$), which implies the marginal benefit of reducing production cost on product variety declines.

PROPOSITION 10. (1) *For a risk-neutral or risk-seeking producer* ($-1 < \Gamma \leq 0$), *the optimal market share is increasing in the production cost reduction*; (2) *for a risk-averse producer* ($\Gamma > 0$) *in a single-segment market*, *the optimal market share is increasing in the production cost reduction*.

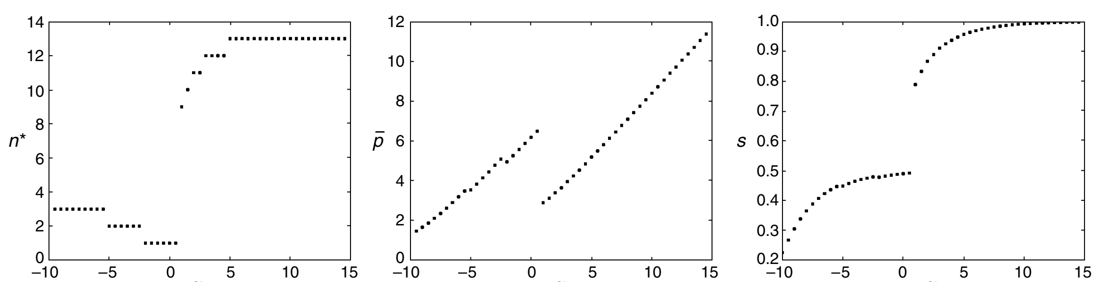
Propositions 9 and 10 show that a decrease in production costs should be exploited by expanding product line length and will result in an increase in market share. Consistent with our results, Kekre and Srinivasan (1990) observed that decreases in production costs accompany increases in market share. They also found a positive impact of market share on profitability and no strong negative impact of product line length on costs. Our results suggest that the positive correlation between market share and profitability is due to the fact that market share is positively correlated with product line length, which indicates low product development and production costs (i.e., high profitability). Because it is unattractive for firms to extend product lines when product development and production costs are high (i.e., product line length is endogenously determined), we would not expect to observe a strong negative impact of product line length on costs. Although Bayus and Putsis (1999) studied product line length and prices endogenously, they did not have direct cost data. However, they did indirectly argue that costs increase with product proliferation, which pushes up price and lowers market share. Again, this is consistent with the predictions of our model.

PROPOSITION 11. *In a single-segment market*, *the optimal price markup is increasing in the production cost reduction*.

In Part 2 of Proposition 9 and Proposition 11, we proved that the optimal product line length and price markup both increase with α in a single-segment market if the producer is risk averse. The following example illustrates that this is not necessarily the case in a market with multiple segments.

EXAMPLE 1. Let $\Gamma = 0.5$, $\beta = 0.05$, $\delta_\mu = 0$ with probability 0.5, and otherwise $\delta_\mu = -10$. We vary α from -10 to 15. The results are shown in Figure 3. The producer originally covers the higher market segment ($\delta_\mu = -10$), which is half of the total market, when $\alpha < 0$. The lower market becomes profitable when α is positive and thus is covered by lowering the price (decreasing the price markup) and offering more variants, which violates Proposition 11. Due to the product development cost and revenue trade-off, there are two ways to transform the production cost saving (increased α) into profit: (1) cutting the product line and saving product design cost and (2) extending the product line and gaining revenue. It turns out in this example that the second choice is preferred when $\alpha > 0$, and the first choice is preferred when $\alpha < 0$. By Proposition 9, we know that the second choice is always better if the producer is risk neutral, even in a multiple-segment market. This example

Figure 3 Optimal Product Line Length, Price Markup, and Market Share for Example 1



demonstrates that if demand is structurally volatile (e.g., from a multisegment market), a risk-averse producer with high production costs may be better off by cutting the product line when the production cost is reduced.

Finally, note that the optimal market share is increasing in α. Under some technical conditions, we are able to prove the monotonicity of the optimal market share with respect to the production cost reduction parameter (α). Because the proof is slightly complicated, we refer the reader to Xu (2005).

Example 1 implies that product variety and pricing decisions in a risk-averse environment are more subtle and unpredictable than in a risk-neutral environment, which provides support for the common management dictum that product development teams should be risk taking (Pearson 1992).

3.4. Risk Coefficient

Proposition 12. *In a single-segment market, the optimal product line length is decreasing in the risk coefficient.*

Proposition 13. *In a single-segment market, the optimal price markup is decreasing in the risk coefficient.*

When a producer becomes risk averse, it will avoid the sunk product development cost and lower prices to attract customers. However, while the optimal product line length and price markup are decreasing in Γ, the following example shows that risk aversion has an indeterminate effect on the optimal market share.

Example 2. Let $S(n) = ne^{-1.6}$ (i.e., $\alpha = -1.6$), $g(n) = n$, and $\beta = 0.01$. Figure 4 shows that optimal product line length and price markup are decreasing, as they must be from Propositions 12 and 13. Hence, when the producer is extremely risk averse, it chooses to quit the market. However we see that market share initially increases to $\Gamma = 2$, as the producer lowers prices to cover more of the market to reduce demand uncertainty, but above this point, as price decreases are forced to slow because of profitability concerns, the decrease in product line length weakens market power, so market share decreases.

3.5. Majorization

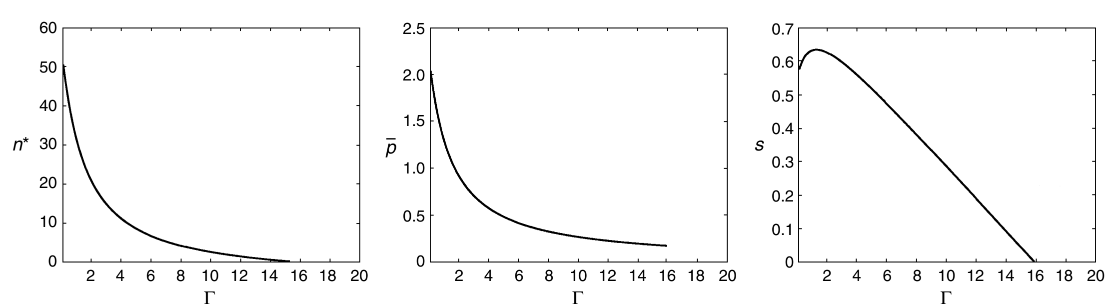
Finally, we can establish a partial order on like-sized menus upon which profit (utility) is monotonic. To do this, we consider two finite menus, M_1 and M_2, which contain n_0 potential variants. Define $\bar{v}^i = \{\bar{v}^i_{(j)}\}_{j=1}^{n_0}$ and $S^i(n)$ ($i = 1, 2$), where $\bar{v}^i_{(j)}$ is the jth best variant value and $S^i(n)$ is product line value for menu i. We say that M_1 is weakly majorized by M_2 if $\sum_{j=1}^{k} \bar{v}^1_{(j)} \leq \sum_{j=1}^{k} \bar{v}^2_{(j)}$, $k = 1, \ldots, n_0$ and that M_1 is majorized by M_2 if $\sum_{j=1}^{n_0} \bar{v}^1_{(j)} = \sum_{j=1}^{n_0} \bar{v}^2_{(j)}$ (see Marshall and Olkin 1979). Let $\pi^*(S^i) = \max_{\bar{p}\geq 0,\, n\geq 0} \pi(\bar{p}, n, S^i(n))$ be the optimal expected profit or utility associated with menu M_i ($i = 1, 2$).

Proposition 14. *The optimal profit/utility is increasing in the order imposed by weak majorization.*

A similar result was developed in van Ryzin and Mahajan (1999, Theorem 3). They call M_1 more *fashionable* than M_2 if M_1 is majorized by M_2. The difference between our majorization and theirs is that our definition considers production costs in addition to consumer preferences (quality indices). Intuitively, the quality markups are "more centralized" on some variants in M_2 than in M_1, which makes it a more profitable offering.

Although Proposition 14 proves that M_2 provides more profit (utility) than M_1, it is possible for the

Figure 4 **Optimal Product Line Length, Price Markup, and Market Share for Example 2**



optimal product line length to be larger under M_2 than under M_1, as we show in the following example.

EXAMPLE 3. Let $S^1 = (1, 2, 3, 4, 4.01, 4.02, \ldots, 4.99, 5)$, $S^2 = (1, 2, 3, 4, 5, 5, 5, \ldots, 5)$, $\beta = 0.05$, and $g(n) = n$. Hence, M_1 is majorized by M_2. For menu M_1, we can compute $n^* = 4$, $\bar{p}^1(n^*) = 1.7178$, and $\pi^1(\bar{p}^1(n^*), n^*) = 0.5178$. For menu M_2, we find that $n^* = 5$, $\bar{p}^2(n^*) = 1.8146$, and $\pi^2(\bar{p}^2(n^*), n^*) = 0.5646$.

EXAMPLE 4. Let $S^1 = (0.01, 0.02, \ldots, 0.99, 1)$, $S^2 = (1, 1, \ldots, 1)$, $\beta = 0.05$, and $g(n) = n$. Again M_1 is majorized by M_2. We can compute that for menu M_1, $n^* = 0$, and for M_2, $n^* = 1$. Hence, the optimal product line length is not monotonic in the majorization partial order of menus.

However, we can show that the optimal product line length *is* monotonic in a more restrictive majorization ordering. Define M_1 to be *monotonically weakly majorized* by M_2 iff M_1 is weakly majorized by M_2 and $\bar{v}^1_{(j)} \geq \bar{v}^2_{(j)}$, $j = 2, \ldots, n_0$. Monotonic weak majorization means quality markups are more centralized on one variant (the one with maximal quality markup) in M_2 than in M_1. Example 3 demonstrates the necessity of this restrictive majorization to guarantee monotonicity in the optimal product line length. Example 4 excludes an uninteresting possibility, that is, a producer with a majorized menu simply quits. We can now show that the optimal product line length is monotonic in the ordering imposed by monotonic weak majorization.

PROPOSITION 15. *In a single-segment market, if a producer with the lowest majorized menu participates, the optimal product line length is decreasing in the ordering imposed by monotonic weak majorization.*

Hence, increasing the attractiveness of the potential menu in an extremely "unbalanced" way, as measured by monotonic weak majorization, causes the producer to offer less variety. This is because individual variants generate more revenue so the firm does not need to offer as many of them. Although M_1 results in more variants than M_2, the following example illustrates that the optimal price markup and market share of M_1 could be either higher or lower than those of M_2, depending on the value of β (the magnitude of design cost).

EXAMPLE 5. Let $g(n) = n$, $S^1 = (3, 6, 6.5)$, and $S^2 = (5, 6.5, 6.5)$, so M_1 is monotonically majorized by M_2.

Table 1 Unit Price Markups and Expected Profits of Menus M_1 and M_2 in Example 5

	n	1	2	3
	$\bar{p}^1$	1.6035	1.8986	1.9369
	$\bar{p}^2$	1.8146	1.9369	1.9369
$\beta = 0.2$	π^1	0.4035	0.4986	0.3369
	π^2	0.6146	0.5369	0.3369
$\beta = 0.1$	π^1	0.5035	0.6986	0.6369
	π^2	0.7146	0.7369	0.6369

The price markups, $\bar{p}^1(n)$ and $\bar{p}^2(n)$, and expected profits, $\pi^1(\bar{p}^1(n), n)$ and $\pi^2(\bar{p}^2(n), n)$, of offering n variants are shown in Table 1.

When $\beta = 0.1$, both menus result in two variants, $\bar{p}^1(2) = 1.8986 < \bar{p}^2(2) = 1.9369$ and the optimal market share $s^1 = 47.33\% < s^2 = 48.37\%$. When $\beta = 0.2$, M_1 still results in two variants, but M_2 results in only one, $\bar{p}^1(2) = 1.8986 > \bar{p}^2(1) = 1.8146$ and $s^1 = 47.33\% > s^2 = 44.89\%$. Hence, while variety is monotonic in the monotonic weak majorization ordering, price and market share may not be.

4. Conclusions and Further Work

In this paper, we study a product line selection and pricing problem. We assume efficient production and distribution systems that imply no economies of scale, but a costly product design process that implies economies of scope and that can be improved by modularity. We see that if a firm reduces the cost of introducing new products (β and K) via modularity, concurrent engineering, or other strategies, it should offer more variety in its product line. Furthermore, because it offers more options to customers, the firm can command a larger market share in a multiple-segment market and charge a price premium (increased price markup) in a single-segment market.

Similarly, if a risk-neutral firm uniformly reduces the variable cost for each product (c_j) via lean manufacturing or other efficiency measures, which reduces the production cost α, it can capitalize on this cost saving by offering more variety, which leads to increases in market share in a multiple-segment market and a price premium in a single-segment market. However, it may be optimal for a risk-averse firm to respond to a reduction in variable cost (i.e., an increase in α) by reducing variety, which demonstrates a fundamental difference in production strategy between a risk-neutral producer and a risk-averse one in a complex market environment.

Finally, in a single-segment market, the more risk averse the firm is, the less variety it should offer, which also implies a lower price margin. We also find monotonicity in product line length by introducing a product assortment order, called monotonic weak majorization. Unfortunately, these results cannot be extended to the multiple-segment market (counterexamples can be found in Xu 2005).

There are several dimensions along which our work could be extended. First, this paper assumes no economies of scale in production and distribution systems and only focuses on the reduction of product development cost by sharing components. Modularity can also be used for postponing differentiation points and inventory pooling (see, e.g., Ramdas 2003 for a summary), which may also reduce overhead cost and affect production costs (a violation of Assumption 3). Explicitly modeling a multiproduct inventory decision involves many technical concerns, such as demand rationing when stockouts occur (see, e.g., Lippman and McCardle 1997, Netessine and Rudi 2003), and hence needs further investigation. Including economies of scale in production would also make the comparison with empirical work (e.g., Kekre and Srinivasan 1990, Bayus and Putsis 1999) more appropriate.

Second, we note that the results of this paper are restricted to monopolistic markets, which also limits the extent to which this work can be compared with some empirical work. A more realistic environment would be a competitive market with variants organized into product platforms. We have taken a step in this direction by considering a duopoly model under the nested-logit demand assumption and have found that most claims still hold for a competitive environment (Hopp and Xu 2005).

Finally, although modularity facilitates product variety, overuse of modularity (especially external modularity) may lead to negative consumer perception (i.e., reducing the quality indices in Assumption 1). If consumers feel variants are undistinguished/redundant or get confused by them, brand equity will suffer (see, e.g., Huffman and Kahn 1998), and hence the modularity strategy could backfire. An extension of our consumer decision model to allow richer interaction between variants would deepen our understanding of the product variety decision problem and the effect of modular design.

Acknowledgments
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Appendix. Proofs

Proof of Proposition 1. Define the price elasticity $\epsilon_{ij} = \partial \ln(r_i)/\partial \ln(\bar{p}_j)$ for all i and j. Because

$$\ln(r_i) = \bar{q}_i - \bar{p}_i + \ln\left(E^{\delta_\mu}\left\{\frac{1}{e^{\delta_\mu} + \sum_{l=1}^n e^{\bar{q}_l - \bar{p}_l}}\right\}\right),$$

it is easy to see that $\epsilon_{i_1 j} = \epsilon_{i_2 j}$ for all $i_1, i_2 \neq j$ and $\epsilon_{jj} = \epsilon_{ij} - \bar{p}_j$ for $i \neq j$. Denote $\epsilon_{\cdot j}$ as the cross-price elasticity, i.e., $\epsilon_{\cdot j} = \epsilon_{lj}$, where $l \neq j$. We check the first-order condition.

When $\gamma = 0$, for $j = 1, \ldots, n$,

$$\begin{aligned}
\frac{\partial E\{\Pi(\bar{\mathbf{p}}, m)\}}{\partial \bar{p}_j} &= N\mu \frac{\partial \sum_{l=1}^n \bar{p}_l r_l}{\partial \bar{p}_j} \\
&= N\mu\left(r_j + \bar{p}_j \frac{\partial r_j}{\partial \bar{p}_j} + \sum_{l\neq j} \bar{p}_l \frac{\partial r_l}{\partial \bar{p}_j}\right) \\
&= N\mu\left(r_j + r_j \epsilon_{jj} + \sum_{l\neq j} \bar{p}_l r_l \frac{\epsilon_{lj}}{\bar{p}_j}\right) \\
&= N\mu\left(r_j + r_j \epsilon_{\cdot j} - \bar{p}_j r_j + \frac{\epsilon_{\cdot j}}{\bar{p}_j} \sum_{l=1}^n \bar{p}_l r_l - r_j \epsilon_{\cdot j}\right).
\end{aligned}$$

Letting $\partial E\{\Pi(\bar{\mathbf{p}}, m)\}/\partial \bar{p}_j = 0$, we have $\bar{p}_j - 1 = (\epsilon_{\cdot j}/r_j \bar{p}_j) \cdot \sum_{l=1}^n \bar{p}_l r_l$, where

$$\frac{\epsilon_{\cdot j}}{r_j \bar{p}_j} = E^{\delta_\mu}\left\{\frac{1}{(e^{\delta_\mu} + \sum_{l=1}^n e^{\bar{q}_l - \bar{p}_l})^2}\right\} \Big/ E^{\delta_\mu}\left\{\frac{1}{e^{\delta_\mu} + \sum_{l=1}^n e^{\bar{q}_l - \bar{p}_l}}\right\}^2,$$

which does not depend on j. Hence, to satisfy the first-order condition, we must have $\bar{p}_j = \bar{p}$, $j = 1, \ldots, n$, where $\bar{p} > 0$.

For $\gamma \neq 0$, recall that $E\{e^{-\gamma \Pi(\bar{\mathbf{p}}, m)}\} = \exp\{N \ln(r_0 + \sum_{j=1}^n r_j e^{-\gamma\mu \bar{p}_j}) + \gamma c_f g(|m|)\}$, so it is equivalent to consider maximizing (or minimizing) $1 + \sum_{j=1}^n r_j (e^{-\gamma\mu \bar{p}_j} - 1)$. For $j = 1, \ldots, n$,

$$\begin{aligned}
&\frac{\partial \sum_{l=1}^n r_l (e^{-\gamma\mu\bar{p}_l} - 1)}{\partial \bar{p}_j} \\
&\quad = -\gamma\mu e^{-\gamma\mu\bar{p}_j} r_j + \frac{\partial r_j}{\partial \bar{p}_j}(e^{-\gamma\mu\bar{p}_j} - 1) + \sum_{l\neq j}(e^{-\gamma\mu\bar{p}_l} - 1)\frac{\partial r_l}{\partial \bar{p}_j} \\
&\quad = -\gamma\mu e^{-\gamma\mu\bar{p}_j} r_j + \epsilon_{jj}\frac{r_j}{\bar{p}_j}(e^{-\gamma\mu\bar{p}_j} - 1) + \frac{\epsilon_{\cdot j}}{\bar{p}_j}\sum_{l\neq j}(e^{-\gamma\mu\bar{p}_l} - 1) r_l \\
&\quad = -\gamma\mu e^{-\gamma\mu\bar{p}_j} r_j - r_j(e^{-\gamma\mu\bar{p}_j} - 1) + \frac{\epsilon_{\cdot j}}{\bar{p}_j}\sum_{l=1}^n (e^{-\gamma\mu\bar{p}_l} - 1) r_l.
\end{aligned}$$

Setting this expression equal to zero, we have

$$(\gamma\mu+1)e^{-\gamma\mu\bar{p}_j} = 1+(\epsilon_{\cdot j}/r_j\bar{p}_j)\sum_{l=1}^{n}(e^{-\gamma\mu\bar{p}_l}-1)r_l.$$

Again, the right-hand side does not depend on j. Hence, to satisfy the first-order condition, we must have $\bar{p}_j=\bar{p}$, $j=1,\ldots,n$, where $\bar{p}>0$.

Finally, note that there is no incentive to offer a variant with negative price markup. □

Note that Proposition 1 holds even if the product development cost function g depends on menu m.

Proof of Proposition 2. It is easy to see that the expected profit and utility increase in $\sum_{j=1}^{n} e^{\bar{q}_j}$. The claim follows immediately from this observation. □

For the remaining proofs, we let R^+ be the set of nonnegative real numbers and Z^+ be the set of nonnegative integers. Although the optimization problem for $\pi(\bar{p},n)$ over n is naturally defined on Z^+, most results derived in this paper hold for both Z^+ and R^+. Therefore, when a property is valid for both cases, we denote it as holding on W^+. To make an approximation on R^+, we define an increasing differentiable version of the product development cost g on R^+ and an increasing, differentiable, and concave version of S on R^+. Note that the optimal number of variants n^* is actually a set, because there may be more than one utility maximizing product line length. If the set n^* contains a singleton, the monotonicity results in this paper have the conventional interpretation; otherwise, they are defined in the sense of set monotonicity, where a collection of subsets of R, $\{X_t\}$ is increasing in t iff $\forall\, t'<t''$, $x_{t'}\in X_{t'}$ and $x_{t''}\in X_{t''}$ imply $\min(x_{t'},x_{t''})\in X_{t'}$ and $\max(x_{t'},x_{t''})\in X_{t''}$. If the greatest (least) element of X_t exists for every t, the greatest (least) element of X_t is increasing in t. Because the conventional calculus approach does not work for comparative statics of optimal sets, we make use of supermodularity theory (see, e.g., Topkis 1998) to prove all results.

As we will see, the interconnection of the parameters (e.g., β, K, and α) and decision variables (n^* and $\bar{p}^*$) is quite complicated but can be captured by means of the *market power index* $\theta=e^{\alpha}S(n)$. We define the market share

$$s(\bar{p},\theta)=E^{\delta_\mu}\left\{\frac{\theta e^{-\bar{p}}}{e^{\delta_\mu}+\theta e^{-\bar{p}}}\right\} \quad \text{and}$$

$$\varphi(\theta)=\begin{cases}\max_{\bar{p}\ge 0} E^{\delta_\mu}\left\{\dfrac{\bar{p}e^{-\bar{p}}}{e^{\delta_\mu}/\theta+e^{-\bar{p}}}\right\}=\max_{\bar{p}\ge 0}\bar{p}\times s(\bar{p},\theta), & \Gamma=0,\\[2ex] \max_{\bar{p}\ge 0} -\dfrac{1}{\Gamma}\ln\left(E^{\delta_\mu}\left\{\dfrac{e^{-(\Gamma+1)\bar{p}}+e^{\delta_\mu}/\theta}{e^{-\bar{p}}+e^{\delta_\mu}/\theta}\right\}\right) & \\[2ex] \quad =\max_{\bar{p}\ge 0} -\dfrac{1}{\Gamma}\ln((e^{-\Gamma\bar{p}}-1)s(\bar{p},\theta)+1), & \Gamma\neq 0,\end{cases}$$

so that $\varphi(\theta)$ represents the optimal revenue if $\Gamma=0$ and $\bar{p}(\theta)$ denotes an optimal price markup for a given market power index θ. Hence, it becomes the optimization problem $\max_{n,\bar{p}}\pi(n,\bar{p})=\max_n \varphi(e^{\alpha}S(n))-\beta g(n)$.

Lemma 1. *For $\Gamma>-1$, $\theta e^{-\bar{p}(\theta)}$ is increasing in θ, $\theta\in W^+$.*

Proof. Let $x=\theta e^{-\bar{p}}$ and $x(\theta)=\theta e^{-\bar{p}(\theta)}$. When $\Gamma=0$,

$$\varphi(\theta)=\max_{\bar{p}\ge 0}E^{\delta_\mu}\left\{\frac{\bar{p}e^{-\bar{p}}}{e^{\delta_\mu}/\theta+e^{-\bar{p}}}\right\}=\max_{0\le x\le\theta}f(x,\theta),$$

$$\text{where}\quad f(x,\theta)=E^{\delta_\mu}\left\{\frac{(\ln\theta-\ln x)x}{e^{\delta_\mu}+x}\right\}.$$

Because

$$\frac{\partial^2}{\partial\theta\partial x}f=\frac{1}{\theta}E^{\delta_\mu}\left\{\frac{e^{\delta_\mu}}{(x+e^{\delta_\mu})^2}\right\}>0,$$

$f(x,\theta)$ is supermodular. Because the set $\{(x,y)\mid y\ge 0,\ y\ge x\ge 0\}$ is a sublattice of W^2, by Theorem 2.8.2 of Topkis (1998), $x(\theta)=\theta e^{-\bar{p}(\theta)}$ increases in θ.

When $0>\Gamma>-1$,

$$\begin{aligned}\varphi(\theta)&=\max_{\bar{p}\ge 0}-\frac{1}{\Gamma}\ln\left(E^{\delta_\mu}\left\{\frac{e^{-(\Gamma+1)\bar{p}}+e^{\delta_\mu}/\theta}{e^{-\bar{p}}+e^{\delta_\mu}/\theta}\right\}\right)\\ &=-\frac{1}{\Gamma}\ln\left(\max_{0\le x\le\theta}f(x,\theta)\right)+\ln(\theta),\end{aligned}$$

$$\text{where}\quad f(x,\theta)=E^{\delta_\mu}\left\{\frac{x^{\Gamma+1}}{x+e^{\delta_\mu}}\right\}+\theta^{\Gamma}E^{\delta_\mu}\left\{\frac{e^{\delta_\mu}}{x+e^{\delta_\mu}}\right\}.$$

Because

$$\frac{\partial^2}{\partial\theta\partial x}f=-\Gamma\theta^{\Gamma-1}E^{\delta_\mu}\left\{\frac{e^{\delta_\mu}}{(x+e^{\delta_\mu})^2}\right\}>0,$$

$f(x,\theta)$ is supermodular. Hence, $x(\theta)=\theta e^{-\bar{p}(\theta)}$ increases in θ.

When $\Gamma>0$, $\varphi(\theta)=(-1/\Gamma)\ln(\min_{0\le x\le\theta}f(x,\theta))+\ln(\theta)$, where f is defined as above. Because $(\partial^2/\partial\theta\partial x)f<0$, $f(x,\theta)$ is submodular. Hence, $x(\theta)=\theta e^{-\bar{p}(\theta)}$ increases in θ for the minimization problem. □

Lemma 2. *$s(\bar{p}(\theta),\theta)$ is increasing in θ.*

Proof. Note that $s(\bar{p},\theta)$ is increasing in $\theta e^{-\bar{p}}$. By Lemma 1, the claim holds. □

Proof of Proposition 3. We prove the result by showing that $\pi(n,\beta)$ is submodular in (n,β), where $\pi(n,\beta)=\max_{\bar{p}\ge 0}\pi(\bar{p},n)$ and $n\in W^+$. This follows by noting that if $\beta'<\beta''$ and $n'<n''$, then $\pi(n',\beta')+\pi(n'',\beta'')-\pi(n',\beta'')-\pi(n'',\beta')=(\beta''-\beta')(g(n')-g(n''))\le 0$, because $g(n)$ is increasing in n. Hence, $\pi(n,\beta)$ is submodular in (n,β). The rest follows from Theorem 2.8.2 of Topkis (1998). □

Proof of Proposition 4. Because $\theta(n)$ increases in n, by Proposition 3 and Lemma 2, the claim holds. □

For a single-segment market ($\delta_\mu=0$),

$$\pi(\bar{p},n)=\frac{e^{\alpha}S(n)\bar{p}e^{-\bar{p}}}{1+e^{\alpha}S(n)e^{-\bar{p}}}-\beta g(n)$$

if $\Gamma=0$;

$$\pi(\bar{p},n)=-\frac{1}{\Gamma}\ln\left(\frac{e^{\alpha}S(n)e^{-(\Gamma+1)\bar{p}}+1}{e^{\alpha}S(n)e^{-\bar{p}}+1}\right)-\beta g(n)$$

if $\Gamma\neq 0$ and $\Gamma>-1$. For fixed n, there exists a unique $\bar{p}(n)$ maximizing $\pi(\bar{p},n)$, where $\bar{p}(n)$ is a concave and increasing function of n and

$$e^{(\Gamma+1)\bar{p}(n)} = \frac{e^{\alpha}S(n)}{(1/\Gamma)(1-(\Gamma+1)e^{-\Gamma\bar{p}(n)})},$$

where $\Gamma = 0$ is interpreted as taking the limit on both sides of the equation. Hence, $\pi(n) = (-1/\Gamma)\ln(\Gamma+1) + \bar{p}(n) - \beta g(n)$. By taking the derivative of $\pi(n)$ and setting it equal to zero, we can show that the set of nonzero optimal product line lengths $n^* \in (R^+)$, the resulting price markup, transformed expected profit/utility, and market share are characterized by

$$\frac{(\Gamma+1)\beta S'(n^*)g'(n^*)}{S'(n^*)-(\Gamma+1)\beta S(n^*)g'(n^*)} = \frac{1}{\Gamma}\left\{e^{\alpha\Gamma}\left[\frac{S'(n^*)-(\Gamma+1)\beta g'(n^*)S(n^*)}{(\Gamma+1)\beta g'(n^*)}\right]^{\Gamma} - 1\right\},$$

$$\bar{p}(n^*) = \alpha + \ln\left(\frac{S'(n^*)-(\Gamma+1)\beta S(n^*)g'(n^*)}{(\Gamma+1)\beta g'(n^*)}\right),$$

$$\pi(n^*) = -\frac{1}{\Gamma}\ln(\Gamma+1) + \bar{p}(n^*) - \beta g(n^*), \quad \text{and}$$

$$s(n^*) = \frac{(\Gamma+1)\beta S(n^*)g'(n^*)}{S'(n^*)}.$$

For details on omitted algebra, see Xu (2005).

Proof of Proposition 5. Note that $1 + e^{\alpha}S(n)e^{-\bar{p}(n)} = (1/\Gamma)(e^{\Gamma\bar{p}(n)} - 1)$, which implies that $\bar{p}(n)$ increases in n, because $S(n)$ is increasing in n. By Proposition 3, $\bar{p}(n^*)$ is decreasing in β. □

Proof of Proposition 6. We prove the result by showing that $\pi(n,K)$ is supermodular in (n,K), where $n \in Z^+$. This follows by noting that if $K_1 < K_2$ and $n_1 < n_2$, then $\pi(n_1,K_1) + \pi(n_2,K_2) - \pi(n_1,K_2) - \pi(n_2,K_1) = -\beta((n_1^{1/K_1} - n_1^{1/K_2}) - (n_2^{1/K_1} - n_2^{1/K_2})) > 0$, because $n^{1/K_1} - n^{1/K_2}$ is increasing in Z^+. Hence, $\pi(n,K)$ is supermodular in (n,K). The rest follows from Theorem 2.8.2 of Topkis (1998). □

Proof of Proposition 7. Similar to the proof of Proposition 4. □

Proof of Proposition 8. Similar to the proof of Proposition 5. □

Lemma 3. *If $0 \geq \Gamma > -1$ and if $E^{\delta_\mu}[e^{-\delta_\mu}] < +\infty$, then $\varphi(\theta) = \varphi(0) + \int_0^\theta \varphi'(s)\,ds$, $\theta \in R^+$.*

Proof. For θ' and $\theta'' \geq 0$, if $\Gamma = 0$,

$$\begin{aligned}
&|\varphi(\theta'') - \varphi(\theta')| \\
&\quad\leq \sup_{\bar{p}\geq 0}\left|E^{\delta_\mu}\left\{\frac{\bar{p}e^{-\bar{p}}}{e^{-\bar{p}}+e^{\delta_\mu}/\theta''}\right\} - E^{\delta_\mu}\left\{\frac{\bar{p}e^{-\bar{p}}}{e^{-\bar{p}}+e^{\delta_\mu}/\theta'}\right\}\right| \\
&\quad= \sup_{\bar{p}\geq 0}\left|\int_{\theta'}^{\theta''} E^{\delta_\mu}\left\{\frac{\bar{p}e^{-\bar{p}}e^{\delta_\mu}}{(e^{\delta_\mu}+\theta e^{-\bar{p}})^2}\right\}d\theta\right| \\
&\quad\leq \int_{\theta'}^{\theta''} \sup_{\bar{p}\geq 0} E^{\delta_\mu}\left\{\frac{\bar{p}e^{-\bar{p}}e^{\delta_\mu}}{(e^{\delta_\mu}+\theta e^{-\bar{p}})^2}\right\}d\theta \\
&\quad\leq \int_{\theta'}^{\theta''} e^{-1}E^{\delta_\mu}[e^{-\delta_\mu}]\,d\theta \\
&\quad= e^{-1}E^{\delta_\mu}[e^{-\delta_\mu}]|\theta' - \theta''|.
\end{aligned}$$

Similarly, if $0 > \Gamma > -1$,

$$\begin{aligned}
&|\varphi(\theta'') - \varphi(\theta')| \\
&\quad\leq \sup_{\bar{p}\geq 0}\left|-\frac{1}{\Gamma}\int_{\theta'}^{\theta''} E^{\delta_\mu}\left\{\frac{(e^{-\Gamma\bar{p}}-1)e^{-\bar{p}}e^{\delta_\mu}}{(e^{\delta_\mu}+\theta e^{-\bar{p}})^2}\right\} \cdot \left[E^{\delta_\mu}\left\{\frac{e^{-(\Gamma+1)\bar{p}}+e^{\delta_\mu}/\theta}{e^{-\bar{p}}+e^{\delta_\mu}/\theta}\right\}\right]^{-1} d\theta\right| \\
&\quad\leq \frac{1}{|\Gamma|}\left|\int_{\theta'}^{\theta''} \sup_{\bar{p}\geq 0} E^{\delta_\mu}\left\{\frac{(e^{-\Gamma\bar{p}}-1)e^{-\bar{p}}e^{\delta_\mu}}{(e^{\delta_\mu}+\theta e^{-\bar{p}})^2}\right\} \cdot \left[E^{\delta_\mu}\left\{\frac{e^{-(\Gamma+1)\bar{p}}+e^{\delta_\mu}/\theta}{e^{-\bar{p}}+e^{\delta_\mu}/\theta}\right\}\right]^{-1} d\theta\right| \\
&\quad\leq \frac{1}{|\Gamma|}\left|\int_{\theta'}^{\theta''} \sup_{\bar{p}\geq 0} E^{\delta_\mu}\left\{\frac{(e^{-\Gamma\bar{p}}-1)e^{-\bar{p}}e^{\delta_\mu}}{(e^{\delta_\mu}+\theta e^{-\bar{p}})^2}\right\} d\theta\right| \\
&\quad\leq \frac{2}{|\Gamma|}E^{\delta_\mu}[e^{-\delta_\mu}]|\theta' - \theta''|.
\end{aligned}$$

Because $\varphi(\theta)$ is absolutely continuous, the result follows. □

The fundamental theorem of integration might not hold for strictly increasing and continuous functions. A counterexample is given in Hewitt and Stromberg (1997, pp. 278–282). Hence, Lemma 3 is necessary for the validity of the fundamental theorem of integration.

Lemma 4. *If $-1 < \Gamma \leq 0$, then $\theta\varphi'(\theta)$ is increasing in $\theta \in \{\theta \geq 0\colon \varphi'(\theta) \text{ exists}\}$.*

Proof. It is easy to check that φ is continuous and strictly increasing in θ. Hence, $\varphi(\theta)$ is differentiable almost everywhere. By Theorem 1 of Milgrom and Segal (2002), at differentiable points,

$$\varphi'(\theta) = \begin{cases} E^{\delta_\mu}\left\{\dfrac{\bar{p}(\theta)e^{-\bar{p}(\theta)}e^{\delta_\mu}}{(e^{\delta_\mu}+\theta e^{-\bar{p}(\theta)})^2}\right\}, & \Gamma = 0, \\[2ex] -\dfrac{1}{\Gamma}E^{\delta_\mu}\left\{\dfrac{e^{-\bar{p}(\theta)}[e^{-\Gamma\bar{p}(\theta)}-1]e^{\delta_\mu}}{(\theta e^{-\bar{p}(\theta)}+e^{\delta_\mu})^2}\right\} \cdot \left[E^{\delta_\mu}\left\{\frac{e^{-(\Gamma+1)\bar{p}(\theta)}+e^{\delta_\mu}/\theta}{e^{-\bar{p}(\theta)}+e^{\delta_\mu}/\theta}\right\}\right]^{-1}, & \Gamma \neq 0 \text{ and } \Gamma > -1. \end{cases}$$

By the first-order condition of the maximization problem over $\bar{p}$, if $\Gamma = 0$,

$$E^{\delta_\mu}\left\{\frac{\bar{p}(\theta)e^{-\bar{p}(\theta)}e^{\delta_\mu}}{(e^{\delta_\mu}/\theta+e^{-\bar{p}(\theta)})^2}\right\} = E^{\delta_\mu}\left\{\frac{\theta e^{-\bar{p}(\theta)}}{e^{\delta_\mu}/\theta+e^{-\bar{p}(\theta)}}\right\},$$

and if $\Gamma \neq 0$,

$$-\frac{1}{\Gamma}E^{\delta_\mu}\left\{\frac{e^{-\bar{p}(\theta)}[e^{-\Gamma\bar{p}(\theta)}-1]e^{\delta_\mu}}{(e^{-\bar{p}(\theta)}+e^{\delta_\mu}/\theta)^2}\right\} = \frac{1}{\Gamma}E^{\delta_\mu}\left\{\frac{\Gamma\theta e^{-(\Gamma+1)\bar{p}(\theta)}}{e^{-\bar{p}(\theta)}+e^{\delta_\mu}/\theta}\right\}.$$

Hence, when $\Gamma=0$,

$$\varphi'(\theta)=E^{\delta_\mu}\left\{\frac{e^{-\bar{p}(\theta)}}{\theta e^{-\bar{p}(\theta)}+e^{\delta_\mu}}\right\},$$

and when $\Gamma\neq 0$,

$$\begin{aligned}\varphi'(\theta)&=E^{\delta_\mu}\left\{\frac{e^{-(\Gamma+1)\bar{p}(\theta)}}{\theta e^{-\bar{p}(\theta)}+e^{\delta_\mu}}\right\}\\&\quad\cdot\left[E^{\delta_\mu}\left\{\frac{\theta e^{-(\Gamma+1)\bar{p}(\theta)}}{\theta e^{-\bar{p}(\theta)}+e^{\delta_\mu}}\right\}+E^{\delta_\mu}\left\{\frac{e^{\delta_\mu}}{\theta e^{-\bar{p}(\theta)}+e^{\delta_\mu}}\right\}\right]^{-1}\end{aligned}$$

at differentiable points.

When $\Gamma=0$,

$$\theta\varphi'(\theta)=E^{\delta_\mu}\left\{\frac{\theta e^{-\bar{p}(\theta)}}{\theta e^{-\bar{p}(\theta)}+e^{\delta_\mu}}\right\},$$

so by Lemma 1, the result follows immediately.

When $-1<\Gamma<0$,

$$\varphi(\theta)=-\frac{1}{\Gamma}\ln\left\{E^{\delta_\mu}\left\{\frac{\theta e^{-(\Gamma+1)\bar{p}(\theta)}}{\theta e^{-\bar{p}(\theta)}+e^{\delta_\mu}}\right\}+E^{\delta_\mu}\left\{\frac{e^{\delta_\mu}}{\theta e^{-\bar{p}(\theta)}+e^{\delta_\mu}}\right\}\right\}.$$

Then,

$$\begin{aligned}\theta\varphi'(\theta)&=E^{\delta_\mu}\left\{\frac{\theta e^{-(\Gamma+1)\bar{p}(\theta)}}{\theta e^{-\bar{p}(\theta)}+e^{\delta_\mu}}\right\}\\&\quad\cdot\left[E^{\delta_\mu}\left\{\frac{\theta e^{-(\Gamma+1)\bar{p}(\theta)}}{\theta e^{-\bar{p}(\theta)}+e^{\delta_\mu}}\right\}+E^{\delta_\mu}\left\{\frac{e^{\delta_\mu}}{\theta e^{-\bar{p}(\theta)}+e^{\delta_\mu}}\right\}\right]^{-1}\\&=1-e^{\Gamma\varphi(\theta)}E^{\delta_\mu}\left\{\frac{e^{\delta_\mu}}{\theta e^{-\bar{p}(\theta)}+e^{\delta_\mu}}\right\}.\end{aligned}$$

By Lemma 1, $E^{\delta_\mu}\{e^{\delta_\mu}/(\theta e^{-\bar{p}(\theta)}+e^{\delta_\mu})\}$ decreases in θ. Because $\Gamma<0$ and $\varphi(\theta)$ increases in θ, the result follows. □

Proof of Proposition 9. We prove the result by showing that $\pi(n,\alpha)$ is supermodular in (n,α), where $\pi(n,\alpha)=\max_{\bar{p}\geq 0}\pi(\bar{p},n)$ and $n\in W^+$.

For Part 1, recall that $\theta=e^{\alpha}S(n)$. Note that $\pi(n,\alpha)=\varphi(e^{\alpha}S(n))-\beta g(n)$. Let $\alpha_l<\alpha_r$ and $n_l<n_r$. Then,

$$\begin{aligned}&\pi(n_r,\alpha_r)+\pi(n_l,\alpha_l)-\pi(n_l,\alpha_r)-\pi(n_r,\alpha_l)\\&\quad=\varphi(S(n_r)e^{\alpha_r})+\varphi(S(n_l)e^{\alpha_l})-\varphi(S(n_r)e^{\alpha_l})-\varphi(S(n_l)e^{\alpha_r})\\&\quad=\int_{S(n_l)e^{\alpha_r}}^{S(n_r)e^{\alpha_r}}\varphi'(s)\,ds-\int_{S(n_l)e^{\alpha_l}}^{S(n_r)e^{\alpha_l}}\varphi'(s)\,ds\\&\quad=\int_{S(n_l)}^{S(n_r)}\varphi'(se^{\alpha_r})e^{\alpha_r}\,ds-\int_{S(n_l)}^{S(n_r)}\varphi'(se^{\alpha_l})e^{\alpha_l}\,ds\\&\quad=\int_{S(n_l)}^{S(n_r)}[\varphi'(se^{\alpha_r})e^{\alpha_r}s-\varphi'(se^{\alpha_l})e^{\alpha_l}s]\frac{1}{s}\,ds\geq 0,\end{aligned}$$

where the last inequality follows from Lemma 4 and the fact that $S(n)$ is increasing in n. Hence, $\pi(n,\alpha)$ is supermodular. Part 1 follows from Theorem 2.8.2 of Topkis (1998).

For Part 2, recall that

$$\pi(n,\alpha)=(-1/\Gamma)\ln(\Gamma+1)+\bar{p}(n,\alpha)-\beta g(n).$$

It is sufficient to prove that

$$(\partial^2/\partial\alpha\partial n)\pi(n,\alpha)=(\partial^2/\partial\alpha\partial n)\bar{p}(n,\alpha)\geq 0.$$

We already have $e^{\alpha}S(n)\Gamma e^{-(\Gamma+1)\bar{p}(n,\alpha)}=1-(\Gamma+1)e^{-\Gamma\bar{p}(n,\alpha)}$. By taking derivatives, it is easy to derive

$$\begin{aligned}e^{\alpha}S'(n)&=(\Gamma+1)[e^{\alpha}S(n)+e^{\bar{p}(n,\alpha)}]\frac{\partial\bar{p}(n,\alpha)}{\partial n},\\e^{\alpha}S(n)&=(\Gamma+1)[e^{\alpha}S(n)+e^{\bar{p}(n,\alpha)}]\frac{\partial\bar{p}(n,\alpha)}{\partial\alpha},\quad\text{and}\\e^{\alpha}S'(n)&=(\Gamma+1)[e^{\alpha}S(n)+e^{\bar{p}(n,\alpha)}]\frac{\partial^2\bar{p}(n,\alpha)}{\partial\alpha\partial n}\\&\quad+(\Gamma+1)\left[e^{\alpha}S'(n)+e^{\bar{p}(n,\alpha)}\frac{\partial\bar{p}(n,\alpha)}{\partial n}\right]\frac{\partial\bar{p}(n,\alpha)}{\partial\alpha}.\end{aligned}$$

By substituting the first-order partial derivatives into the last equation and simplifying it, we obtain

$$\frac{\partial^2\bar{p}(n,\alpha)}{\partial\alpha\partial n}=\frac{e^{\alpha}S'(n)e^{(\Gamma+2)\bar{p}(n,\alpha)}}{(\Gamma+1)^2(e^{\alpha}S(n)+e^{\bar{p}(n,\alpha)})^3}\geq 0.$$

Hence, $\pi(n,\alpha)$ is supermodular. Part 2 also follows from Theorem 2.8.2 of Topkis (1998). □

Proof of Proposition 10. Similar to the proof of Proposition 4. □

Proof of Proposition 11. Similar to the proof of Proposition 5. □

Proof of Proposition 12. We prove the result by showing that $\pi(n,\Gamma)$ is submodular in (n,Γ), where $\pi(n,\Gamma)=\max_{\bar{p}\geq 0}\pi(\bar{p},n)$ and $n\in W^+$. Because

$$\pi(n,\Gamma)=(-1/\Gamma)\ln(\Gamma+1)+\bar{p}(n,\Gamma)-\beta g(n)$$

and

$$(\partial^2/\partial\Gamma\partial n)\pi(n,\Gamma)=(\partial^2/\partial\Gamma\partial n)\bar{p}(n,\Gamma),$$

it is sufficient to prove that $(\partial^2/\partial\Gamma\partial n)\bar{p}(n,\Gamma)\leq 0$. Recall that $S(n)\Gamma e^{-(\Gamma+1)\bar{p}}=1-(\Gamma+1)e^{-\Gamma\bar{p}}$. By taking derivatives of both sides, we obtain

$$\begin{aligned}S'(n)&=(\Gamma+1)\frac{\partial\bar{p}(n,\Gamma)}{\partial n}[S(n)+e^{\bar{p}(n,\Gamma)}],\\S(n)e^{-\bar{p}(n,\Gamma)}&-S(n)\Gamma e^{-\bar{p}(n,\Gamma)}\bar{p}(n,\Gamma)+1-(\Gamma+1)\bar{p}(n,\Gamma)\\&=(\Gamma+1)\Gamma[S(n)e^{-\bar{p}(n,\Gamma)}+1]\frac{\partial\bar{p}(n,\Gamma)}{\partial\Gamma},\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial^2\bar{p}(n,\Gamma)}{\partial n\partial\Gamma}\\&\quad=-\frac{\frac{\partial\bar{p}(n,\Gamma)}{\partial n}[S(n)+e^{\bar{p}(n,\Gamma)}]+(\Gamma+1)e^{\bar{p}(n,\Gamma)}\frac{\partial\bar{p}(n,\Gamma)}{\partial n}\frac{\partial\bar{p}(n,\Gamma)}{\partial\Gamma}}{(\Gamma+1)[S(n)+e^{\bar{p}(n,\Gamma)}]}\\&\quad=\left\{1+\frac{e^{2\bar{p}(n,\Gamma)}}{\Gamma[S(n)+e^{\bar{p}(n,\Gamma)}]^2}\right.\\&\qquad\left.\cdot\left[S(n)e^{-\bar{p}(n,\Gamma)}-S(n)\Gamma\bar{p}(n,\Gamma)e^{-\bar{p}(n,\Gamma)}+1-(\Gamma+1)\bar{p}(n,\Gamma)\right]\right\}\end{aligned}$$

$$\times \frac{-S'(n)}{(\Gamma+1)^2[S(n)+e^{\bar{p}(n,\Gamma)}]}$$
$$= \frac{-S'(n)e^{(2+\Gamma)\bar{p}(n,\Gamma)}}{(\Gamma+1)^2\Gamma^2[S(n)+e^{\bar{p}(n,\Gamma)}]^3}\left[e^{\Gamma\bar{p}(n,\Gamma)}-1-\Gamma\bar{p}(n,\Gamma)\right]\le 0,$$

where the last equality is substituting

$$S(n)=\frac{1}{\Gamma}e^{(\Gamma+1)\bar{p}(n,\Gamma)}-\left(1+\frac{1}{\Gamma}\right)e^{\bar{p}(n,\Gamma)}$$

and the last inequality follows from $S'(n)\ge 0$ and the fact that e^x-x-1 is minimized with the value of zero when $x=0$. The result follows from Theorem 2.8.2 of Topkis (1998). □

Proof of Proposition 13. We already have shown that $\bar{p}(n,\Gamma)$ increases in n for fixed Γ. By Proposition 12, the optimal product line length decreases in Γ. Hence, it is sufficient to prove that $\partial\bar{p}(n,\Gamma)/\partial\Gamma\le 0$. Recall that

$$(\Gamma+1)[S(n)e^{-\bar{p}(n,\Gamma)}+1]\frac{\partial\bar{p}(n,\Gamma)}{\partial\Gamma}$$
$$=\frac{1}{\Gamma}S(n)e^{-\bar{p}(n,\Gamma)}-S(n)e^{-\bar{p}(n,\Gamma)}\bar{p}(n,\Gamma)+\frac{1}{\Gamma}-\left(\frac{1}{\Gamma}+1\right)\bar{p}(n,\Gamma)$$
$$=\frac{e^{\Gamma\bar{p}(n,\Gamma)}}{\Gamma^2}[1-e^{-\Gamma\bar{p}(n,\Gamma)}-\Gamma\bar{p}(n,\Gamma)]\le 0,$$

because $1-e^{-x}-x\le 0$. □

Proof of Proposition 14. We assume that M_1 is weakly majorized by M_2. Recall that

$$\pi(\bar{p},n,S^i(n))=E^{\delta_\mu}\left\{\frac{S^i(n)\bar{p}e^{-\bar{p}}}{e^{\delta_\mu}+S^i(n)e^{-\bar{p}}}\right\}-\beta g(n)$$

if $\Gamma=0$. Otherwise, if $\Gamma\ne 0$ and $\Gamma>-1$,

$$\pi(\bar{p},n,S^i(n))=-\frac{1}{\Gamma}\ln\left(E^{\delta_\mu}\left\{\frac{S^i(n)e^{-(\Gamma+1)\bar{p}}+e^{\delta_\mu}}{S^i(n)e^{-\bar{p}}+e^{\delta_\mu}}\right\}\right)-\beta g(n),$$

where $i=1,2$. Because $\bar{v}^1$ is weakly majorized by $\bar{v}^2$, $S^1(n)\le S^2(n)$, $n=1,\ldots,n_0$. Hence, $\pi(\bar{p},n,S^1(n))\le\pi(\bar{p},n,S^2(n))$, where $n=1,\ldots,n_0$. The result follows immediately. □

Lemma 5. *Assume that $\delta_\mu=0$, $dS^1(n)/dn\ge dS^2(n)/dn$ for $n\ge 1$ and $n\in R^+$, and that M_1 is weakly majorized by M_2, i.e., $S^2(n)\ge S^1(n)$. Then, $\arg\max_{n\ge 1,\,n\in W^+}\pi(n,M_i)$ is decreasing in i, where $i=1,2$.*

Proof. We construct a third menu M_λ by $S_\lambda(n)=S^1(n)+\lambda(S^2(n)-S^1(n))$, where $\lambda\in[0,1]$,

$$\pi(\bar{p},n,\lambda)=\frac{S_\lambda(n)\bar{p}e^{-\bar{p}}}{1+S_\lambda(n)e^{-\bar{p}}}-\beta g(n),$$

and $\pi(n,\lambda)=\max_{\bar{p}\ge 0}\pi(\bar{p},n,\lambda)$. The menu M_λ serves as a "bridge" connecting M_1 and M_2; as λ goes from 0 to 1, M_1 transitions to M_2 smoothly.

For $\Gamma=0$, recall that $\pi(n,\lambda)=\max_{\bar{p}\ge 0}\pi(\bar{p},n,\lambda)=\bar{p}(n,\lambda)-1-\beta g(n)$.

$$e^{\bar{p}(n,\lambda)}=\frac{S_\lambda(n)}{\bar{p}(n,\lambda)-1}\quad\text{implies}\quad\bar{p}(n,\lambda)e^{\bar{p}(n,\lambda)}\frac{\partial\bar{p}(n,\lambda)}{\partial n}=\frac{\partial S_\lambda(n)}{\partial n}$$

and

$$\bar{p}(n,\lambda)e^{\bar{p}(n,\lambda)}\frac{\partial\bar{p}(n,\lambda)}{\partial\lambda}=\frac{\partial S_\lambda(n)}{\partial\lambda}.$$

By taking the derivative with respect to λ, we obtain

$$\frac{\partial^2 S_\lambda(n)}{\partial\lambda\partial n}=\bar{p}(n,\lambda)e^{\bar{p}(n,\lambda)}\frac{\partial^2\bar{p}(n,\lambda)}{\partial n\partial\lambda}+\frac{\partial\bar{p}(n,\lambda)}{\partial n}\frac{\partial\bar{p}(n,\lambda)}{\partial\lambda}(1+\bar{p}(n,\lambda))e^{\bar{p}(n,\lambda)}.$$

Because

$$\frac{\partial^2 S_\lambda(n)}{\partial\lambda\partial n}=\frac{dS^2(n)}{dn}-\frac{dS^1(n)}{dn}\le 0,$$
$$\frac{\partial S_\lambda(n)}{\partial\lambda}=S^2(n)-S^1(n)\ge 0,\quad\text{and}$$
$$\frac{\partial S_\lambda(n)}{\partial n}=(1-\lambda)\frac{dS^1(n)}{dn}+\lambda\frac{dS^2(n)}{dn}\ge 0,$$
$$\bar{p}(n,\lambda)e^{\bar{p}(n,\lambda)}\frac{\partial^2\bar{p}(n,\lambda)}{\partial n\partial\lambda}=\frac{\partial^2 S_\lambda(n)}{\partial\lambda\partial n}-\frac{\partial S_\lambda(n)}{\partial n}\frac{\partial S_\lambda(n)}{\partial\lambda}\frac{1+\bar{p}(n,\lambda)}{\bar{p}(n,\lambda)^2e^{\bar{p}(n,\lambda)}}\le 0.$$

For $\Gamma\ne 0$, recall that $S_\lambda(n)\Gamma e^{-(\Gamma+1)\bar{p}(n,\lambda)}=1-(\Gamma+1)\cdot e^{-\Gamma\bar{p}(n,\lambda)}$. By taking derivatives on both sides of the above equation, we obtain

$$\frac{\partial S_\lambda(n)}{\partial n}=(\Gamma+1)\frac{\partial\bar{p}(n,\lambda)}{\partial n}[S_\lambda(n)+e^{\bar{p}(n,\lambda)}],$$
$$\frac{\partial S_\lambda(n)}{\partial\lambda}=(\Gamma+1)\frac{\partial\bar{p}(n,\lambda)}{\partial\lambda}[S_\lambda(n)+e^{\bar{p}(n,\lambda)}],\quad\text{and}$$
$$\frac{\partial^2 S_\lambda(n)}{\partial n\partial\lambda}=(\Gamma+1)\left[\left(\frac{\partial S_\lambda(n)}{\partial\lambda}+e^{\bar{p}(n,\lambda)}\frac{\partial\bar{p}(n,\lambda)}{\partial\lambda}\right)\frac{\partial\bar{p}(n,\lambda)}{\partial n}+(S_\lambda(n)+e^{\bar{p}(n,\lambda)})\frac{\partial^2\bar{p}(n,\lambda)}{\partial n\partial\lambda}\right].$$

Hence,

$$(\Gamma+1)(S_\lambda(n)+e^{\bar{p}(n,\lambda)})\frac{\partial^2\bar{p}(n,\lambda)}{\partial n\partial\lambda}=\frac{dS^2(n)}{dn}-\frac{dS^1(n)}{dn}-(S^2(n)-S^1(n))\cdot\left(1+\frac{e^{\bar{p}(n,\lambda)}}{(\Gamma+1)(S_\lambda(n)+e^{\bar{p}(n,\lambda)})}\right)\frac{\partial S_\lambda(n)/\partial n}{S_\lambda(n)+e^{\bar{p}(n,\lambda)}}\le 0,$$

because $dS^1(n)/dn\ge dS^2(n)/dn$, $S^2(n)\ge S^1(n)$, and $S_\lambda(n)$ is increasing in n.

Hence, $\partial^2\bar{p}(n,\lambda)/\partial n\partial\lambda\le 0$. By Theorem 2.8.2 of Topkis (1998), the claim follows. □

Proof of Proposition 15. We assume that M_1 is weakly majorized by M_2 and a producer with menu M_1 participates. By Proposition 14, a producer with menu M_2 participates also. Hence, we need only focus on $n\ge 1$ and $n\in Z^+$.

Define $S^i(n)$ as the piecewise linear interpolation that connects $\{(n, \sum_{j=1}^{n} \bar{v}_{(j)}^{M_i})\}_{n=1}^{n_0}$, $i = 1, 2$. Because M_1 is monotonically weakly majorized by M_2,

$$\frac{dS^1(n)}{dn} = \bar{v}_{(j)}^{M_1} \geq \frac{dS^2(n)}{dn} = \bar{v}_{(j)}^{M_2} \quad \forall\, n \in [j-1, j).$$

By Lemma 5, the result follows. □

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