CUSTOMER HETEROGENEITY AND STRATEGIC BEHAVIOR IN REVENUE MANAGEMENT: A MARTINGALE APPROACH

XIAOWEI XU AND WALLACE J. HOPP

DEPARTMENT OF INDUSTRIAL ENGINEERING

NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208, USA

Abstract. This paper studies the effects of customer heterogeneity and strategic behavior on dynamic pricing strategies. We assume that heterogenous customers arrive during a fixed season according to a Poisson process with intensity as a function of price. We obtain an implicit optimal pricing policy by solving a HJB equation. Our results show that optimal prices follow a submartingale (supermartingale) if price sensitivity is constant or decreasing (increasing) in time. We also demonstrate martingale properties for the arrival intensity and consumer surplus processes. Finally, we build heterogenous customer profiles, composed of value of time and price sensitivity, and show that an increasing (decreasing) value of time function results in increasing (decreasing) prices and decreasing (increasing) customer price sensitivity over the course of the season.

Date: November 2, 2004.
Email: xuxiaowe@northwestern.edu and hopp@northwestern.edu.

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1. Introduction and Literature Review

Since advances in information technology have brought capability for companies to track sales promptly and adjust pricing levels rapidly, revenue management (RM) has been broadly adopted in the airline, retailing and other industries and extensively studied by academic researchers in the past decade (see Bitran and Caldentey 2003 and Elmaghraby and Keskinocak 2003 for reviews). Most of the existing literature focuses on algorithmic issues of finding optimal pricing policies given exogenous input data (e.g., demand modes) and has paid little attention to classification of business situations in which different types of pricing strategies are suited. For instance, airline ticket prices often increase as the flight date approaches, but retail goods are usually marked down as the end of the season approaches.

In this paper, we propose a framework for understanding how the physical characteristics of customers and goods/services impact the pricing strategy of an industry. This framework differs from those used in most RM papers by characterizing aggregate customer behavior by means of an endogenous dynamic price sensitivity function, rather than using an exogenous demand process. This framework also enables us to address the important question of how strategic customer behavior impacts pricing strategies. This has significant practical consequences because as customers are continuously being “trained”, retailers are finding it increasingly difficult to lure customers into stores before the “last minute” (see, e.g., Wall Street Journal 11/23/2002). Many, such as auto dealers, have had to resort to large monetary incentives to attract customers (see, e.g., Wall Street Journal 1/6/2004). However, as pointed out by Elmaghraby and Keskinocak (2003) and Talluri and Van Ryzin (2004), strategic customer behavior has been largely overlooked in the academic literature, largely because it is difficult to model.

This paper studies the effects of customer heterogeneity and strategic behavior on dynamic pricing from an analytical point of view. We adopt the model setup of Gallego and van Ryzin (1994). Customer arrivals are modeled by a Poisson process with intensity as a function of price. Customers are heterogenous in price sensitivity, which varies over time. The resulting intensity control problem
can be solved via a HJB equation (Bremaud 1980). We focus on three demand modes: exponential, linear and iso-elastic. Although closed form solutions are rare (except for exponential demand with constant price sensitivity), a recursive structure exists that enables us to develop probabilistic properties of the optimal prices, arrival intensity and consumer surplus.

Our main tool is martingale theory and Dynkin’s formula (Rogers and Williams 1987), which have been used in obtaining optimal expected revenue functions (see, e.g., Feng and Xiao (2000) and Feng and Gallego (2000)). Xu and Hopp (2004a, 2004b) also applied martingale theory to study probabilistic properties of optimal prices and consumer surplus, but their approach is based on induction and only works for exponential demand with responsive pricing. Our results show that optimal prices follow a submartingale if price sensitivity is constant or decreasing in time, but optimal prices become a supermartingale if price sensitivity is increasing. This property is robust over various demand modes. We also demonstrate martingale properties for the intensity and consumer surplus processes. These results are consistent with some previous results that characterized price trends due to demand learning effects. For instance, Lazear (1986) showed a decrease in price due to the sequential learning of consumer reservation values. Pashigian (1988) generalized Lazear’s model to an equilibrium setting. Xu and Hopp (2004b) showed a markdown due to the sequential learning of a downward demand trend.

Strategic customer behavior can play an important role in determining an appropriate dynamic pricing policy. For example, Besanko and Winston (1990) studied a price skimming problem without inventory/capacity constraint. They found that ignoring strategic customer behavior can seriously reduce profit. Modeling strategic customer behavior can be controversial because it often hinges on an assumption of customer rational expectation (see, e.g., Stokey 1979). In the RM literature on dynamic pricing with constrained inventory/capacity, customers not only need to form rational expectation of price paths but must also form beliefs about the amount of inventory/capacity—an extremely demanding behavioral assumption. For this reason, some authors have assumed limited information on the part of customers. For instance, Elmaghraby et al. (2002) studied the effect
of strategic customer behavior in a multiple unit auction, in which the number of units for sale is assumed to be common knowledge. Unlike the seller, who often has the ability to monitor everything and make rational decisions, customers lack the ability to monitor price paths and make complicated inferences. In this paper, we follow this approach of limiting customer knowledge to an initial information set. Specifically, we assume that customers commit at the outset to their purchasing time given their rational expectation of prices and beliefs about the initial inventory amount.

As noted by Talluri and Van Ryzin (2004), modeling strategic customer behavior is essentially a mechanism design problem. In this paper, we model a customer’s utility function as the difference between the value of time and rational expectation of prices. Customers are heterogeneous in their value of time functions and price sensitivity. We derive a form of the value of time function, which is similar to the form of the payoff function in the revenue equivalence theorem in mechanism design (see, e.g., Krishna 2002), such that no strategic customer behavior occurs.

The remainder of the paper is organized as follows. Section 2 presents a deterministic model. We study the stochastic dynamic pricing model in Section 3. Section 4 examines a model of strategic customer behavior. Finally we discuss our conclusions and future research topics in Section 5. All omitted proofs are given in the Appendix.

2. A DETERMINISTIC INTERTEMPORAL RESOURCE ALLOCATION MODEL

We assume that a firm has limited stock $C$ of a single good or service with which to fulfill demand over the interval $[0, T]$. The demand rate at time $s \in [0, T]$ is known and is denoted by $\lambda(p, s)$, where $p$ is the price at time $s$. Define the dynamic price $p(\lambda, s)$ as the inverse of $\lambda(p, s)$ and the revenue rate $r(\lambda, s) = \lambda p(\lambda, s)$, which is assumed concave in $\lambda$. The firm maximizes revenue $\int_0^T q(s)\lambda(q(s), s)ds$ such that $\int_0^T \lambda(q(s), s)ds \leq C$, where $q(s)$ is a pricing policy. This problem was studied in Bitran and Caldentey (2003). The optimal condition is $p^*(s) = \mu - \lambda(p^*(s), s)/\partial r(\lambda^*(s), s)/\partial p$ or equivalently $\partial r(\lambda^*(s), s)/\partial \lambda = \nu$, where $\mu \geq 0$, $\lambda^*(s) = \lambda(p^*(s), s)$ and $s \in [0, T]$. Hence, dynamic pricing is essentially a tool for allocating a limited resource over time in a way that achieves equal marginal revenue.
Notice that if \( \lambda(p, s) = \chi(p)g(s) \), i.e., \( \lambda(p, s) \) can be decomposed into a function of price \( \chi(p) \) and a function of time \( g(s) \), then \( p^*(s) = \mu - \chi(p^*(s))\frac{\partial \chi(p^*(s))}{\partial p} \), which implies a constant optimal price (Gallego and van Ryzin 1994). Xu and Hopp (2004a) found that the optimal prices follow a martingale, provided the seller uses responsive pricing to sell all of her inventory and demand is iso-elastic with a random multiplicative term. Unfortunately, many demand functions cannot be decomposed into a product of a function of price and a function of time, as we demonstrate in the following examples. For simplicity, we assume \( \mu > 0 \) in the following examples.

**Example 1.** Suppose the demand rate is exponential, given by \( \lambda(p, s) = e^{-\alpha(s)p} \), where \( \alpha(s) \) is a positive function, and hence consumer surplus is \( v(p, s) = \frac{1}{\alpha(s)} e^{-\alpha(s)p} \). It is easy to see that the price elasticity at time \( s \) is \( \alpha(s)p \). Then price, sales, and consumer surplus are given by \( p^*(s) = \mu + \frac{1}{\alpha(s)} \), \( \lambda^*(s) = e^{-1-\mu\alpha(s)} \) and \( v^*(s) = v(p^*(s), s) = \frac{1}{\alpha(s)} e^{-1-\mu\alpha(s)} \), respectively. If \( \alpha(s) \) is constant (increasing, decreasing), then \( p^*(s), \lambda^*(s) \) and \( v^*(s) \) are constant (decreasing, increasing). We conclude that the seller should sell more at higher prices when customers are less price sensitive.

**Example 2.** Suppose the demand rate is linear, given by \( \lambda(p, s) = a - \alpha(s)p \), where \( p \in [0, \frac{a}{\alpha(s)}] \) and \( a > 0 \), and hence consumer surplus is \( v(p, s) = \frac{(a - \alpha(s)p)^2}{2\alpha(s)} = \frac{\lambda^2(p, s)}{2\alpha(s)} \). Then \( p^*(s) = 0.5(\mu + \frac{a}{\alpha(s)}) \), \( \lambda^*(s) = 0.5(a - \mu\alpha(s)) \), \( v^*(s) = \frac{(a - \mu\alpha(s))^2}{8\alpha(s)} \), where we assume \( \mu < \frac{a}{\sup_s \alpha(s)} \). If \( \alpha(s) \) is constant (increasing, decreasing), then \( p^*(s) \) and \( \lambda^*(s) \) are constant (decreasing, increasing). If \( \alpha(s) \) is constant, \( v^*(s) \) is constant, but \( v^*(s) \) may not be monotonic in \( s \) if \( \alpha(s) \) is increasing or decreasing.

**Example 3.** Suppose the demand rate is iso-elastic, given by \( \lambda(p, s) = p^{-\alpha(s)} \), where \( \alpha(s) > 1 \), and hence consumer surplus is \( v(p, s) = \frac{p^{1-\alpha(s)}}{\alpha(s)-1} \). Then \( p^*(s) = \mu(1 + \frac{1}{\alpha(s)-1}) \), \( \lambda^*(s) = (\frac{\mu a(s)}{\alpha(s)-1})^{-\alpha(s)} \) and \( v^*(s) = (\frac{\mu a(s))^{-\alpha(s)}}{(\alpha(s)-1)^{1-\alpha(s)}} \). If \( \alpha(s) \) is constant (increasing, decreasing), then \( p^*(s) \) is constant (decreasing, increasing). If \( \alpha(s) \) is constant, \( \lambda^*(s) \) and \( v^*(s) \) are constant, but \( \lambda^*(s) \) and \( v^*(s) \) may not be monotonic in \( s \) if \( \alpha(s) \) is increasing or decreasing.
3. A Stochastic Model with three demand modes

We now consider a stochastic version of the model in Section 2. We assume that a firm has a stock level of \( n \) (a nonnegative integer) at time 0 and sells them over the interval \([0, T]\). Consumer demand follows a non-homogenous Poisson process \( M_q(s) \) with intensity \( \lambda(q(s), s) \), where \( q(s) \) is a non-anticipatory pricing policy that satisfies \( \int_0^T dM_q(s) \leq n \). At time \( s \), with remaining inventory level \( l \), the firm maximizes expected revenue \( J_q(l, s) = E[\int_s^T q(t)dM_q(t)] \), where \( J_q(l, T) = 0 \) and \( J_q(0, s) = 0 \) for \( l \leq n \) and \( s \in [0, T] \), by finding an optimal pricing policy \( p^*(l, s) \in \arg \max_q J_q(l, s) \). We denote the optimal expected revenue in \([s, T]\) given inventory level \( l \) by \( J^*(l, s) = J_{p^*}(l, s) = \sup_q J_q(l, s) \).

Let \( I^*(l, s) = J^*(l, s) - J^*(l - 1, s) \), where \( 1 \leq l \leq n \) and \( J^*(0, s) = 0 \). By Theorem T1 in Chapter VII of Bremaud (1980), we obtain the Hamilton-Jacobi sufficient conditions for \( J^* \),

\[
-\frac{\partial J^*(l, s)}{\partial s} = \sup_{\lambda} [r(\lambda) - \lambda I^*(l, s)] , \quad 1 \leq l \leq n \quad \text{and} \quad s \in [0, T].
\]

There exists a unique optimal solution \( \lambda^*(l, s) \) for the right side of the above equation, since \( r(\lambda) \) is concave in \( \lambda \). Let \( p^*(l, s) = p(\lambda^*(l, s), s) \).

Although \( p^*, \lambda^* \) and \( J^* \) must be computed numerically for most cases (Stadje 1990), there exists an implicit structure among \( p^*, \lambda^* \) and \( J^* \), which enables us to prove a martingale property for the optimal price process by applying Dynkin’s formula (Rogers and Williams 1987).

Let \( N(s) \) be a nonhomogeneous Markovian process, which models the inventory process driven by the optimal pricing policy \( p^* \). Let \( N(0) = n \) and define the intensity matrix \( Q(s) = (q_{i,j}(s)) \), where \( q_{i,i-1}(s) = \lambda^*(i, s), q_{i,i}(s) = -q_{i,i-1}(s) \) and \( q_{0,j}(s) = 0 \) for \( 1 \leq i \leq n \) and \( 0 \leq j \leq n \). Let \( g(i, s) \) be absolutely continuous with respect to \( s \) for \( 0 \leq i \leq n \). Then by Dynkin’s formula (Rolski et al. 1999, p453), \( G(s) \) is a \( \mathcal{F}^N_s \)-martingale, where \( G(s) = g(N(s), s) - \int_0^s \frac{\partial g}{\partial s}(N(t), t) dt + \sum_{j \neq N(t)} q_{N(t), j}(t)(g(j, t) - g(N(t), t)) dt \) and \( \mathcal{F}^N_s \) is a \( \sigma \)-algebra generated by \( \{N(t), t \leq s\} \).

Define a stopping time \( \tau = \inf\{s \in [0, T] | N(s) \leq 1\} \), which represents the time when inventory level falls to one. Let a price process \( P(N(s), s) = p^*(N(s), s) \) if \( N(s) \geq 1 \) and \( P(0, s) = p_0 \) if \( N(s) = 0 \), where \( p_0 \) is a constant. Then \( P(N(s \wedge \tau), s \wedge \tau) \) coincides with the optimal price path before the inventory level \( N(s) \) hits one and stays flat after that.
In the following sections, we study the martingale property of $P(N(s \wedge \tau), s \wedge \tau)$ under three demand modes: exponential, linear and iso-elastic.

3.1. The case of exponential demand. We define an exponential demand rate $\lambda(p, s) = ae^{-\alpha(s)p}$, where $a$ is a positive constant and $\alpha(s)$ is positive and continuously differentiable in $s$. Hence, the price process is $p(\lambda, s) = \frac{1}{\alpha(s)} \ln(\frac{\lambda}{\lambda})$ and the revenue rate is $r(\lambda, s) = \frac{\lambda}{\alpha(s)} \ln(\frac{\lambda}{\lambda})$. The consumer surplus at time $s$ is $v(p, s) = \frac{a}{\alpha(s)} e^{-\alpha(s)p}$. Solving the Hamilton-Jacobi sufficient conditions for $J^*$ yields $\lambda^*(l, s) = \lambda e^{-I^*(l, s)/\alpha(s)}$, where $\lambda = ae^{-1}$ and $l \geq 1$. Hence, $p^*(l, s) = I^*(l, s) + \frac{1}{\alpha(s)}$ and $\frac{\partial J^*(l, s)}{\partial s} = -\frac{\lambda}{\alpha(s)} e^{-I^*(l, s)/\alpha(s)} = -\frac{\lambda^*(l, s)}{\alpha(s)}$ for $l \geq 1$. Kincaid and Darling (1963), Stadje (1990) and Gallego and van Ryzin (1994) give an exact solution, $J^*(l, s) = \frac{1}{\alpha} \log(\sum_{i=0}^{l} \frac{(\lambda(l-s))^i}{i!})$, for the case $\alpha(s) = \alpha$.

Proposition 1. For an exponential demand process, if $\alpha(s)$ is a constant or decreasing, $P(N(s \wedge \tau), s \wedge \tau)$ is a $\mathcal{F}_{s \wedge \tau}$-submartingale.

As an example, consider air travel where leisure travellers, who are sensitive to price, buy tickets earlier than do business travellers, who are less sensitive to price. Hence, we would expect $\alpha(s)$ to be decreasing in time. Proposition 1 shows that under these conditions, the optimal price path is increasing in a probabilistic sense, which is consistent with empirical observations of airline ticket prices. Unlike the deterministic case (Example 1), since the seller is uncertain whether or not she will be able to sell all her inventory, she tends to set price lower at the beginning than later in the season, even if customers have the same price sensitivity (i.e., $\alpha(s)$ is a constant).

Stadje (1990) studied the case for $\alpha(s) = e^{Ks}$ and $n = 1$. Proposition 1 shows the increase of optimal prices in a probabilistic sense when $K \leq 0$.

Proposition 2. For an exponential demand process, if $\alpha(s) = \alpha e^{Ks}$, where $K \geq 2\lambda$, $P(N(s \wedge \tau), s \wedge \tau)$ is a $\mathcal{F}_{s \wedge \tau}$-supermartingale.
To appreciate the meaning of Proposition 2, consider a fashion goods market. In this environment, latecomers tend to be bargain hunters (e.g., shopping for a swimsuit at the end of the summer) who are price sensitive. Hence, \( \alpha(s) \) is increasing in time. Proposition 2 implies that the optimal price path is decreasing in a probabilistic sense, which is consistent with markdowns. Indeed, this is exactly what we observe in fashion markets.

We define the optimal intensity process as \( \Lambda(N(s), s) = \lambda^*(N(s), s) \) if \( N(s) \geq 1 \), \( \Lambda(0, s) = 0 \) and denote the consumer surplus process by \( V(N(s), s) = \Lambda(N(s), s) \alpha(s) \).

**Proposition 3.** For an exponential demand process, if \( \alpha(s) \) is constant (decreasing, increasing), \( \Lambda(N(s \wedge \tau), s \wedge \tau) \) and \( V(N(s \wedge \tau), s \wedge \tau) \) are \( \mathcal{F}_{s \wedge \tau} \)-martingales (submartingales, supermartingales).

Proposition 3 suggests that the seller should sell more when customers are less price sensitive and customers with less price sensitivity receive more surplus. These conclusions are consistent with our observations in Example 1. When \( \alpha(s) = \alpha \), we can reduce the intensity and consumer surplus to closed form expressions, \( E[\Lambda(N(s \wedge \tau), s \wedge \tau)] = \Lambda(n, 0) = \frac{\lambda \left( \sum_{i=0}^{n-1} \frac{(\lambda T)^i}{i!} \right)}{\sum_{i=0}^{n} \frac{(\lambda T)^i}{i!}} \) and \( E[V(N(s \wedge \tau), s \wedge \tau)] = \frac{\lambda^2 \left( \sum_{i=0}^{n-1} \frac{(\lambda T)^i}{i!} \right)}{\sum_{i=0}^{n} \frac{(\lambda T)^i}{i!}} \), which are increasing in \( n \) by Theorem 1 of Gallego and van Ryzin (1994). Hence, higher inventory at the firm translates into a higher sales rate and surplus to customers.

**Example 4.** We revisit the numerical example in the bottom of Figure 1 in Gallego and van Ryzin (1994). Let \( T = 1 \), \( a = 100 \), \( n = 25 \) and \( \alpha(s) = e^{Ks} \), where \( K = -1, 0, 0.5, 1.5 \). We simulate \( N(s) \) by a discrete time Markov chain (Kushner and Dupuis 2001).

Figure 1 shows the mean and quartiles of the optimal price process \( P(N(s \wedge \tau), s \wedge \tau) \). Figures 1a and 1b show an increase in optimal prices, which is consistent with Proposition 1. Figure 1c shows a decrease in optimal prices, but a short increase at the end; this occurs because \( K = 0.5 \) does not satisfy the condition in Proposition 2. As \( |K| \) becomes larger, the upward or downward trend of optimal prices becomes steeper. Although \( K = 1.5 \) does not satisfy Proposition 2, the mean of the
optimal prices becomes strictly decreasing. However, because optimal prices jump up with a drop in inventory, the quartiles also jump up.

Figure 2 shows the mean and quartiles of the consumer surplus process $V(N(s \land \tau), s \land \tau)$. All figures are consistent with Proposition 3, showing monotonic trends. Notice that, as with price, the trend in consumer surplus becomes steeper as $|K|$ becomes larger.
3.2. The case of linear demand. We define a linear demand rate \( \lambda(p, s) = a - \alpha(s)p \), where \( p \in [0, \frac{a}{\alpha(s)}] \) and \( a > 0 \). Hence, \( p(\lambda, s) = \frac{a - \lambda}{\alpha(s)} \), where \( \lambda \in [0, a] \), and the consumer surplus \( v(p, s) = \frac{(a - \alpha(s)p)^2}{2\alpha(s)} \). We solve the Hamilton-Jacobi sufficient conditions for \( J^* \), which gives \( \lambda^*(l, s) = 0.5(a - \alpha(s)I^*(l, s)) \), \( p^*(l, s) = 0.5(\frac{a}{\alpha(s)} + I^*(l, s)) \) and \( \frac{\partial J^*(l, s)}{\partial s} = -\frac{(a - \alpha(s)I^*(l, s))^2}{4\alpha(s)} = -\frac{\lambda^*(l, s)^2}{\alpha(s)} \), where \( s \in [0, T] \) and \( 1 \leq l \leq n \). For this demand function, we assume that \( \alpha(s) \) is increasing or constant. Hence, \( I^*(l, s) = J^*(l, s) - J^*(l - 1, s) \leq \sup_{t \in [s, T]} \frac{a}{\alpha(t)} \leq \frac{a}{\alpha(s)} \) and \( \lambda^*(l, s) \) and \( p^*(l, s) \) fall into the feasible regions. If \( \alpha(s) \) is decreasing, \( I^*(l, s) \) may be larger than \( \frac{a}{\alpha(s)} \) and then \( \lambda^*(l, s) \) and \( p^*(l, s) \) are outside the feasible regions, which complicates analysis. Hence, we only consider the case of constant or increasing \( \alpha(s) \).

**Proposition 4.** For a linear demand process, if \( \alpha(s) \) is a constant, \( P(N(s \wedge \tau), s \wedge \tau) \) is a \( \mathcal{F}_{s \wedge \tau}^N \)-submartingale.

**Proposition 5.** For a linear demand process, if \( \alpha(s) = \alpha e^{Ks} \), where \( K \geq a \), \( P(N(s \wedge \tau), s \wedge \tau) \) is a \( \mathcal{F}_{s \wedge \tau}^N \)-supermartingale.

These results are identical to those for the exponential demand case. Recall that the optimal intensity process is \( \Lambda(N(s), s) = \lambda^*(N(s), s) \) if \( N(s) \geq 1 \), \( \Lambda(0, s) = 0 \) and the consumer surplus process is \( V(N(s), s) = \frac{\lambda^2(N(s), s)}{2\alpha(s)} \).

**Proposition 6.** For a linear demand process, if \( \alpha(s) \) is constant or increasing, \( \Lambda(N(s \wedge \tau), s \wedge \tau) \) is a \( \mathcal{F}_{s \wedge \tau}^N \)-supermartingale.

This is not identical to the exponential demand case, for which if \( \alpha(s) \) is constant the optimal intensity process is a martingale (Proposition 3). Hence, the martingale property of the optimal intensity process is sensitive to demand mode. Finally, we can show:

**Proposition 7.** For a linear demand process, if \( \alpha(s) \) is constant (increasing), \( V(N(s \wedge \tau), s \wedge \tau) \) is a \( \mathcal{F}_{s \wedge \tau}^N \)-martingale (supermartingale).
This agrees with Proposition 3 and shows that unlike the deterministic case of Example 2, the consumer surplus process shows monotonic probabilistic trends in the stochastic case.

3.3. The case of iso-elastic demand. We define an iso-elastic demand rate $\lambda(p, s) = p^{-\alpha(s)}$, where $\alpha(s) > 1$. Hence, $p(\lambda, s) = \lambda^{\frac{1}{\alpha(s)}}$ and the consumer surplus $v(p, s) = \frac{p^{1-\alpha(s)}}{\alpha(s)-1}$. We solve the Hamilton-Jacobi sufficient conditions for $J^*$, which gives $\lambda^*(l, s) = (\alpha(s)-1)\frac{\alpha(s)I^*(l, s)^{-\alpha(s)}}{\alpha(s)^{\alpha(s)}}$, $p^*(l, s) = \frac{\alpha(s)I^*(l, s)}{\alpha(s)^{\alpha(s)}}$ and $\frac{\partial J^*(l, s)}{\partial s} = -\frac{(\alpha(s)-1)\alpha(s)^{\alpha(s)-1}}{\alpha(s)^\alpha(s)}I^*(l, s)^{1-\alpha(s)}$, where $s \in [0, T]$ and $1 \leq l \leq n$.

Proposition 8. For an iso-elastic demand process, if $\alpha(s)$ is a constant or decreasing, $P(N(s \wedge \tau), s \wedge \tau)$ is a $F^{N}_{s \wedge \tau}$-submartingale.

Proposition 8 shows an increasing trend of prices even if $\alpha(s)$ is constant. In contrast, Xu and Hopp (2004a) found that if the seller employs responsive-pricing to make certain that all inventories are sold, the optimal prices follow a martingale when $\alpha(s)$ is constant.

Recall that the consumer surplus process $V(N(s), s) = v(P(N(s), s), s)$ if $N(s) \geq 1$ and $V(0, s) = 0$.

Proposition 9. For an iso-elastic demand process, if $\alpha(s)$ is constant, $V(N(s \wedge \tau), s \wedge \tau)$ is a $F^{N}_{s \wedge \tau}$-martingale.

Notice that Proposition 9 agrees with Proposition 7 and is consistent with Example 3. In contrast, Xu and Hopp (2004a) found that the consumer surplus process follows a submartingale in a constant elasticity responsive-pricing setting since the price shows a flat trend (a martingale).

In a summary, we have shown that the martingale properties of the optimal price process and consumer surplus process are robust to demand modes, but that the optimal intensity process is not. It is easy to see that the probabilistic monotonic trends of optimal prices, intensity and consumer surplus exist locally when $\alpha(s)$ is locally monotonic, since the local monotonicity of $\alpha(s)$ gives a local increase/decrease of the compensator and the global arguments in all proofs are directly applicable to proving the corresponding local properties.
Figure 3. The mean of optimal price process $P(N(s \wedge \tau), s \wedge \tau)$. (a) $\alpha(s) = e^{(s-t_c)^+}$; (b) $E[P(N(s \wedge \tau), s \wedge \tau)]$.

Example 5. Let $T = 1$ and $n = 25$. We consider linear demand $\lambda(p,s) = a - \alpha(s)p$, where $a = 40$, $\alpha(s) = e^{(s-t_c)^+}$ and $t_c \in \{0.6, 0.8, 0.9\}$. As shown in Figure 3(a), the price sensitivity is flat until $s = t_c$ and then increases. Figure 3(b) shows that price is almost flat with a slight increase until $s = t_c$, which coincides with Proposition 4, and then drops sharply (suggested by Proposition 5), which represents markdowns due to a shift in customer groups from fashion pursuers to bargain hunters.

4. A MODEL OF CUSTOMER BEHAVIOR

In the previous section, we assumed that $\alpha(t)$ is exogenous. In reality, strategic customers may form rational expectation of price trends and hold back (or rush) their purchases in hopes of an extraordinary deal, which may skew the assumed shape of $\alpha(t)$. In this section, we study a stylized model to identify customer decision processes and preferences, under which $\alpha(t)$ is endogenously determined and stable with respect to strategic customer behavior.

We label every customer by her price sensitivity $\alpha(s)$. We assume at time 0 customers correctly form a rational expectation of prices $RP(t; \alpha, n) = E[P(N(t \wedge \tau), t \wedge \tau)]$, which is generated by the revenue maximization of a firm given $\alpha(t)$ and initial inventory amount $n$. We assume that $RP(t; \alpha, n)$ is strictly increasing (strictly decreasing) in $t$ if $\alpha(t)$ is decreasing (increasing) in $t$, that
is, \(\alpha'(t)RP'(t;\alpha,n) < 0\), which is valid for many of the cases in Section 3. Since customers may not be informed of the inventory amount \(n\), we assume a customer with price sensitivity \(\alpha(s)\) has a subjective belief \(\pi_{\alpha(s)}\) on \(n\). Let \(RP(t;\alpha,\pi_{\alpha(s)}) = \int RP(t;\alpha,n)\pi_{\alpha(s)}(dn)\). We assume that a customer with price sensitivity \(\alpha(s)\) values time by \(VT(t;\alpha,\beta(\alpha(s)),\pi_{\alpha(s)}) = \frac{1}{\alpha^{\beta(\alpha(s))-1}(s)} \int_0^t \alpha^{\beta(\alpha(s))}(z)RP'(z;\alpha,\pi_{\alpha(s)})dz\), where \(\beta\) is a positive function of \(\alpha(s)\). Hence, a customer with price sensitivity \(\alpha(s)\) has utility \(u(t;\alpha(s)) = VT(t;\alpha,\beta(\alpha(s)),\pi_{\alpha(s)}) - \alpha(s)RP(t;\alpha,\pi_{\alpha(s)}),\ t \in [0,T]\). Notice that if \(\alpha(t)\) is constant, then \(u(t;\alpha(s))\) is constant in \(t\).

If \(\alpha(t)\) is increasing (decreasing), the value of time is decreasing (increasing) in \(t\), since \(RP'(t;\alpha,\pi_{\alpha(s)}) < 0\) (> 0). For instance, fashion pursuers value early purchase of new products, which implies a decrease of \(V(t;\alpha,\beta(\alpha(s)),\pi_{\alpha(s)})\) in \(t\) for all \(s \in [0,T]\). However, affordability considerations may delay some of their purchases, which implies an increasing trend in \(\alpha(t)\). Another example is that of travellers who value delaying purchase of airline tickets until the last moment in order to avoid disruptions from unexpected schedule changes, which implies an increase of \(V(t;\alpha,\beta(\alpha(s)),\pi_{\alpha(s)})\) in \(t\) for all \(s \in [0,T]\). However, business travellers (leisure travellers) tend to purchase late (early) given their low (high) sensitivity of prices, which implies a decreasing trend in \(\alpha(t)\).

Now, suppose that first, every customer commits to her purchasing time \(t(\alpha(s))\) ex ante by maximizing \(u(t;\alpha(s))\). It is easy to see that \(\frac{\partial u(t;\alpha(s))}{\partial t} = \frac{\alpha^{\beta(\alpha(s))(t) - \alpha^{\beta(\alpha(s))}(s)}{\alpha^{\beta(\alpha(s))-1}(s)} \alpha'(t)RP'(t;\alpha,\pi_{\alpha(s)})\) and \(\frac{\partial^2 u(t;\alpha(s))}{\partial^2 t} = \frac{\beta(\alpha(s))\alpha^{\beta(\alpha(s))}(s)\alpha'(t)RP'(t;\alpha,\pi_{\alpha(s)})}{\alpha^{\beta(\alpha(s))-1}(s)} + \frac{\alpha^{\beta(\alpha(s))(t) - \alpha^{\beta(\alpha(s))}(s)}{\alpha^{\beta(\alpha(s))-1}(s)} \alpha''(t)RP''(t;\alpha,\pi_{\alpha(s)})\). The first order condition gives a unique global maximum \(t(\alpha(s)) = s\), since \(\frac{\partial^2 u(t;\alpha(s))}{\partial t^2}\big|_{t=s} = \beta(\alpha(s))\alpha'(s)RP'(s;\alpha,\pi_{\alpha(s)}) < 0\).

Second, at time \(s\) a customer with price sensitivity \(\alpha(s)\) receives a random shock \(\varepsilon\), which represents individual heterogeneity, and purchases only if \(-\alpha(s)p + \varepsilon > 0\) (the reservation value is positive), where \(p\) is the price at time \(s\). If \(\varepsilon\) is exponentially distributed, we have the exponential demand rate \(e^{-\alpha(s)p}\) of Section 3.1. If \(\varepsilon\) is uniform on \([0,a]\), we have the linear demand rate of Section 3.2.
Given the above customer decision process and preference over time and price, the firms’ profit maximization does not induce any strategic customer behavior and keeps the shape of $\alpha(t)$ unchanged. It is easy to see that the quasi-linear form of utility functions performs a similar simplification in determining $V(t; \alpha, \beta(\alpha(s)), \pi(\alpha(s)))$ as it does in determining the payoff function in the revenue equivalence theorem in mechanism design (see, e.g., Krishna 2002). By checking the first and second order conditions, we see that $V(t; \alpha, \beta(\alpha(s)), \pi(\alpha(s)))$ satisfies incentive compatibility constraints, i.e., a customer with price sensitivity $\alpha(s)$ will purchase at time $s$. Notice that if $\beta(s) = 1$ and $\pi(\alpha(s)) = \pi$, then $V(t; \alpha, \beta(\alpha(s)), \pi(\alpha(s)))$ is type-independent. The following proposition shows its uniqueness.

**Proposition 10.** There exists an unique value of time function, $VT(t; \alpha) = \int_0^t \alpha(z)RP'(z; \alpha, \pi)dz$, which is type-independent.

If we have data on the value of time for different customer segments, denoted by $\{f(t; \gamma), \gamma \in \Gamma\}$, where $f$ is the value of time for customers with price sensitivity $\gamma$, then by curve fitting, we can estimate a class of $(\hat{\alpha}, \hat{\beta}, \hat{\pi})$ such that $f(t; \gamma) \approx VT(t; \alpha, \beta(\gamma), \pi(\gamma))$ and $\gamma = \alpha(s)$, where $s, t \in [0, T]$ and $\gamma \in \Gamma$. This gives us a set of implementable candidates for $\alpha(s)$. Since the set could be large, we need marketing data to filter out the right $\hat{\beta}$ and a communication process (e.g., through advertising) to make customers focus on one $\hat{\pi}$. Notice that in Proposition 10, if $f(t; \gamma)$ is increasing (decreasing), then $RP(z; \alpha, \pi)$ is increasing (decreasing), which implies $\alpha(s)$ is decreasing (increasing). Hence, the utility change of a physical attribute of a product/service over time determines the price trend and the trend of customer heterogeneity in customer arrivals.

In another direction, we can also recover $f(t; \gamma)$ by $VT(t; \alpha, \beta(\alpha(s)), \pi(\alpha(s)))$ and $\gamma = \alpha(s)$ given $\alpha$ and $\beta$. The following example demonstrates this procedure.

**Example 6.** Let $T = 1$, $a = 100$, $\alpha(s) = e^{Ks}$ and $\beta(\alpha(s)) = 1$, where $K \in \{-1, 1.5\}$. We consider an exponential demand function and vary $n \in \{10, 25, 50\}$. Figure 4(a) shows the rational expectation of prices $RP(t)$ and Figure 4(b) shows the corresponding value of time function $VT(t)$ for the case of $K = -1$. Although $RP(t)$ varies significantly in parallel, $VT(t)$ is very similar.
When $\alpha(s)$ is increasing, as shown in Figure 5, the rational expectation of prices and value of time function decrease in time. The value of time function $VT(t)$ is sensitive to the accuracy of information on the initial inventory level. Since different beliefs result in various shapes of $VT(t)$, if $\alpha(s)$ is endogenously determined, the shape of $\alpha(s)$ may be very different.

5. Conclusions and future research

In this paper, we study the effects of customer heterogeneity and strategic behavior on dynamic pricing strategies. We assume that heterogenous customers arrive according to a Poisson process
with intensity as a function of price. Our results show that optimal prices follow a submartingale if price sensitivity is constant or decreasing in time, but become a supermartingale if price sensitivity is increasing. This property is robust to demand modes (exponential, linear and iso-elastic), as shown in Table 1, where $\uparrow$ (-, $\downarrow$, $\updownarrow$) represents either an increasing (flat, decreasing, indeterminate) trend or a submartingale (martingale, supermartingale, neither). We also demonstrate martingale properties for the arrival intensity and consumer surplus processes.

We also adopt a mechanism design approach to study strategic customer behavior. By assuming that a customer’s utility function is given by the difference between the value of time and rational expectation of prices, we derive a form of the value of time function, which is part of heterogenous customer profiles (the other part is price sensitivity). We show that price trends and customer heterogeneity in arrivals are endogenously determined by the valuation change of physical attributes of a product/service over time.

Finally, we compute an optimal state-dependent pricing policy for the revenue management problem under various customer arrival and demand modes. We also determine the mode endogenously by solving an open loop (state-independent) customer decision process. This characterization is appropriate for situations where customers do not change their purchasing decisions based on price observations (e.g., business travellers buy airline tickets close to their departure dates regardless of short-term changes in pricing, such as promotions).
In some situations, however, customers may monitor prices, at least occasionally, and henceforth their purchasing timing may be state dependent. To obtain an optimal state-dependent policy for timing purchases would require building a complete dynamic Bayesian game and finding a perfect Nash equilibrium. This would involve tremendous complication of modeling and analysis issues and thus seems almost impossible in the current dynamic and stochastic setting. But an interesting research question is whether or not we can build and solve a complete dynamic Bayesian game in a simpler setting.

Some other research questions are also intriguing. First, in practice prices do not change continuously, but are instead restricted to various points (e.g., $10.99, $9.99, etc). Recognizing this, Gallego and van Ryzin (1994) examined a setting in which prices can change within a set containing a few points. Hence, an interesting question is whether limiting the price range reduces strategic customer behavior. Second, we know that demand learning also affects price trends. Therefore, combining demand learning (see, e.g., Lazear 1986, Aviv and Pazgal 2002 and Xu and Hopp 2004b) and customer heterogeneity would give us a more complete understanding of the factors that affect dynamic pricing strategies.

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Appendix: Proofs

Lemma 1. For an exponential demand process, if \( \alpha(s) \) is a constant or decreasing and \( p_0 \geq \sup_{s \in [0,T]} J^*(1,s) + \frac{2}{\alpha(T)} \), \( P(N(s),s) \) is a \( \mathcal{F}_s \)-submartingale.

Proof. By Dynkin’s formula, we only need to prove \( C(i,t) = \frac{\partial P}{\partial s}(i,t) + \sum_{j \neq i} q_{i,j}(t)(P(j,t) - P(i,t)) > 0 \) for all \( t \in [0,T] \) and \( 0 \leq i \leq n \). Since \( P(0,s) = p_0 \), which is a constant, and \( q_{0,j}(t) = 0 \), \( C(0,t) = 0 \).

Recall that \( p^*(i,t) = I^*(i,t) + \frac{1}{\alpha(t)} \) and \( \frac{\partial J^*(i,t)}{\partial s} = -\frac{\lambda^*(i,t)}{\alpha(t)} \) for \( i \geq 1 \). \( C(1,t) = -\frac{\alpha'(t)}{\alpha^2(t)} + \frac{\partial J^*(1,t)}{\partial s} + \lambda^*(1,t)(p_0 - J^*(1,t) - \frac{1}{\alpha(t)}) \geq \lambda^*(1,t)(p_0 - J^*(1,t) - \frac{2}{\alpha(t)}) \geq 0 \). For \( 2 \leq i \leq n \), \( C(i,t) = -\frac{\alpha'(t)}{\alpha^2(t)} + \frac{\partial J^*(i,t)}{\partial s} + \lambda^*(i,t)(p_0 - J^*(i,t) - \frac{1}{\alpha(t)}) \geq \lambda^*(i,t)(p_0 - J^*(i,t) - \frac{2}{\alpha(t)}) \geq 0 \).
Lemma 3. follows from Proof. Lemma 2.

Proof. For 2 $t \in [0, T]$ and 0 $i \leq n$. It is easy to see that $C(0, t) = 0$ and $C(1, t) \leq -\lambda^*(1, t)(J^*(1, t) + \frac{2}{\alpha(t)}) \leq 0$. For $2 \leq i \leq n$, $C(i, t) = -\frac{\alpha'(t)}{\alpha^2(t)} + \frac{\lambda^*(i, t)}{\alpha(t)} (e^{\alpha(t)(I^*(i, t) - I^*(i-1, t))} - 1 - \alpha(t)(I^*(i, t) - I^*(i-1, t)))) \leq -\frac{\alpha'(t)}{\alpha^2(t)} + \frac{2\lambda^*(i, t)}{\alpha(t)} e^{\alpha(t)(I^*(i, t) - I^*(i-1, t))} \leq \frac{1}{\alpha(t)}(-K + 2\lambda) \leq 0$, where the second to the last inequality follows from $\lambda^*(i, t) = \lambda e^{-I^*(i, t)\alpha(t)}$ and $I^*(i-1, t) \geq 0$.

Proof of Proposition 2.

Proof. By Lemma 2 and the optional sampling theorem of martingales (Karatzas and Shreve 1988).

Lemma 2. For an exponential demand process, if $\alpha(s) = \alpha e^{Ks}$, where $K \geq 2\lambda$, and $p_0 = 0$, $P(N(s), s)$ is a $\mathcal{F}_s^N$-supermartingale.

Proof. The proof is similar to the proof of Lemma 1. We only need to show that $C(i, t) \leq 0$ for all $t \in [0, T]$ and $0 \leq i \leq n$. It is easy to see that $C(0, t) = 0$ and $C(1, t) \leq -\lambda^*(1, t)(J^*(1, t) + \frac{2}{\alpha(t)}) \leq 0$. For $2 \leq i \leq n$, $C(i, t) = -\frac{\alpha'(t)}{\alpha^2(t)} + \frac{\lambda^*(i, t)}{\alpha(t)} (e^{\alpha(t)(I^*(i, t) - I^*(i-1, t))} - 1 - \alpha(t)(I^*(i, t) - I^*(i-1, t)))) \leq -\frac{\alpha'(t)}{\alpha^2(t)} + \frac{2\lambda^*(i, t)}{\alpha(t)} e^{\alpha(t)(I^*(i, t) - I^*(i-1, t))} \leq \frac{1}{\alpha(t)}(-K + 2\lambda) \leq 0$, where the second to the last inequality follows from $\lambda^*(i, t) = \lambda e^{-I^*(i, t)\alpha(t)}$ and $I^*(i-1, t) \geq 0$.

Proof of Proposition 3.

Proof. By Lemma 2 and the optional sampling theorem of martingales (Karatzas and Shreve 1988).

Lemma 3. For an exponential demand process, if $\alpha(s)$ is constant (decreasing, increasing) and $\delta > 0$, $\Delta(N(s), s) = \frac{\Lambda(N(s), s)}{\alpha^*(s)}$ is a $\mathcal{F}_s^N$-martingale (submartingale, supermartingale).

Proof. By Dynkin’s formula, we only need to prove $C(i, t) = \frac{\partial\Delta}{\partial s}(i, t) + \sum_{j \neq i} q_{i,j}(t)(\Delta(j, t) - \Delta(i, t)) = 0$ ($\geq 0$, $\leq 0$) for all $t \in [0, T]$ and $0 \leq i \leq n$ if $\alpha(s)$ is constant (decreasing, increasing). It is easy to see that $C(0, t) = 0$. For $i \geq 1$, $C(i, t) = -\frac{\delta\Lambda(i,t)\alpha'(t)}{\alpha^*(i,t)} + \Lambda(i,t) (-I^*(i, t)\alpha'(t) - \alpha(t)\frac{\partial I^*(i,t)}{\partial s}) + \lambda^*(i, t)(\Delta(i-1, t) - \Delta(i, t)) = \Delta(i, t)(-\frac{\delta\alpha'(t)}{\alpha(t)} - I^*(i, t)\alpha'(t) + \Lambda(i, t) - \Lambda(i-1, t) + \Lambda(i, t)(\frac{\Lambda(i-1,t)-1}{\Lambda(i,t)-1}) = -\alpha'(t)\Delta(i, t)(\frac{\delta}{\alpha(t)} + I^*(i, t)))$. The rest is similar to proof of Lemma 1.
Proof of Proposition 3.

Proof. By Lemma 3 and the optional sampling theorem of martingales (Karatzas and Shreve 1988). □

Lemma 4. For a linear demand process, if $\alpha(s) = \alpha$ and $p_0 \geq 0.25 \sup_{s \in [0,T]} J^*(1,s) + \frac{0.75\alpha}{\alpha}$, $P(N(s), s)$ is a $\mathcal{F}_s^N$-submartingale.

Proof. By Dynkin’s formula, we only need to prove $C(i, t) = \frac{\partial P}{\partial s}(i, t) + \sum_{j \neq i} q_{i,j}(t)(P(j, t) - P(i, t)) \geq 0$ for all $t \in [0, T]$ and $0 \leq i \leq n$. Since $P(0, s) = p_0$, which is a constant, and $q_{0,j}(t) = 0$, $C(0, t) = 0$. Recall that $p^*(i, t) = 0.5\left(\frac{a}{\alpha} + I^*(i, t)\right)$ and $\frac{\partial J^*(i,t)}{\partial s} = -\frac{\lambda^*(i,t)}{\alpha}$ for $i \geq 1$. $C(1, t) = 0.5\frac{\partial J^*(1,t)}{\partial s} + \lambda^*(1,t)(p_0 - 0.5J^*(1,t) - \frac{0.5\alpha}{\alpha}) = \lambda^*(1,t)(p_0 - 0.5J^*(1,t) - \frac{0.5\alpha}{\alpha} - \frac{0.5\lambda^*(1,t)}{\alpha}) = \lambda^*(1,t)(p_0 - 0.25J^*(1,t) - \frac{0.75\alpha}{\alpha}) \geq 0$. For $2 \leq i \leq n$, $C(i, t) = 0.5\frac{\partial J^*(i,t)}{\partial s} - 0.5\frac{\partial J^*(i-1,t)}{\partial s} + 0.5\lambda^*(i,t)(I^*(i-1,t) - I^*(i,t)) \geq \frac{(\lambda^*(i,t) - \lambda^*(i-1,t))^2}{2\alpha} \geq 0$. As the compensator is an increasing process, the claim follows immediately. □

Proof of Proposition 4.

Proof. By Lemma 4 and the optional sampling theorem of martingales (Karatzas and Shreve 1988). □

Lemma 5. For a linear demand process, if $\alpha(s) = \alpha e^{Ks}$, where $K \geq a$, and $p_0 = 0$, $P(N(s), s)$ is a $\mathcal{F}_s^N$-supermartingale.

Proof. The proof is similar to the proof of Lemma 4. We only need to show that $C(i, t) \leq 0$ for all $t \in [0, T]$ and $0 \leq i \leq n$. It is easy to see that $C(0, t) = 0$ and $C(1, t) = -\lambda^*(1,t)(0.25J^*(1,t) + \frac{0.75\alpha}{\alpha(t)}) \leq 0$. For $2 \leq i \leq n$, $C(i, t) = -\frac{0.5a}{\alpha^2(t)} + \frac{(\lambda^*(i,t) - \lambda^*(i-1,t))^2}{2\alpha(t)} \leq -\frac{aK}{2\alpha(t)} + \frac{a^2}{2\alpha(t)} \leq 0$, where the second to the last inequality follows from $\lambda^*(i,t) \leq a$. □

Proof of Proposition 5.

Proof. By Lemma 5 and the optional sampling theorem of martingales (Karatzas and Shreve 1988). □
Lemma 6.

Proof. By Dynkin’s formula, we only need to check the sign of $C(i, t) = \frac{\partial V}{\partial s}(i, t) + \sum_{j \neq i} q_{i,j}(t)(\Lambda(j, t) - \Lambda(i, t))$ for all $t \in [0, T]$ and $0 \leq i \leq n$. It is obvious that $C(0, t) = 0$. For $i \geq 1$, $C(i, t) = -0.5\alpha'(t)I^*(i, t) - 0.5\alpha(t)(\frac{\partial J^*(i, t)}{\partial s} - \frac{\partial J^*(i-1, t)}{\partial s}) + \lambda^*(i, t)(\Lambda(i-1, t) - \Lambda(i, t)) = -0.5\alpha'(t)I^*(i, t) - 0.5(\Lambda(i-1, t)^2 - \Lambda(i, t)^2) + \Lambda(i, t)(\Lambda(i-1, t) - \Lambda(i, t)) = -0.5\alpha'(t)I^*(i, t) - 0.5(\Lambda(i-1, t)^2 - \Lambda(i, t)^2)$.

If $\alpha(s)$ is constant or increasing, $C(i, t) \leq 0$. Hence, $\Lambda(N(s), s)$ is a $\mathcal{F}^N_s$-supermartingale. By the optional sampling theorem of martingales (Karatzas and Shreve 1988), the claim holds.

Proof of Proposition 7.

Proof. The proof is similar to prove of Proposition 6. We define $C(i, t) = \frac{\partial V}{\partial s}(i, t) + \sum_{j \neq i} q_{i,j}(t)(V(j, t) - V(i, t))$. It is easy to see that $C(0, t) = 0$. For $i \geq 1$, $C(i, t) = -\frac{\Lambda(i, t)^2\alpha'(t)}{2\alpha(t)^2} + \frac{\Lambda(i, t)}{\alpha(t)}(-0.5\alpha'(t)I^*(i, t) - 0.5(\Lambda(i-1, t)^2 - \Lambda(i, t)^2)) + \lambda^*(i, t)(V(i-1, t) - V(i, t)) = -\frac{\Lambda(i, t)^2\alpha'(t)}{2\alpha(t)^2} - \frac{\alpha'(i, t)\Lambda(i, t)}{2\alpha(t)} - \Lambda(i, t)(V(i-1, t) - V(i, t)) + \Lambda(i, t)(V(i-1, t) - V(i, t)) = -\alpha'(t)(\frac{\Lambda(i, t)^2}{2\alpha(t)^2} + \frac{J^*(i, t)\Lambda(i, t)}{2\alpha(t)})$. If $\alpha(s)$ is constant (increasing), $C(i, t) = 0 (\leq 0, \geq 0)$, which implies that $V(N(s), s)$ is a martingale (supermartingale). By the optional sampling theorem of martingales (Karatzas and Shreve 1988), the claim holds.

Lemma 6. For an iso-elastic demand process, if $\alpha(s)$ is a constant or decreasing and

$$ p_0 \geq \sup_{s \in [0, T]} \left( \frac{\alpha(s)}{\alpha(s)-1} \right)^2 J^*(1, s), \quad P(N(s), s) \text{ is a } \mathcal{F}^N_s \text{-submartingale.} $$

Proof. By Dynkin’s formula, we only need to prove $C(i, t) = \frac{\partial P}{\partial s}(i, t) + \sum_{j \neq i} q_{i,j}(t)(P(j, t) - P(i, t)) \geq 0$ for all $t \in [0, T]$ and $0 \leq i \leq n$. Since $P(0, s) = p_0$, which is a constant, and $q_{0,j}(t) = 0$, $C(0, t) = 0$. Recall that $\lambda^*(i, t) = (\frac{\alpha(t)-1}{\alpha(t)})^{\alpha(t)}I^*(i, t)\alpha^{-\alpha(t)}$, $p^*(i, t) = \frac{\alpha(t)}{\alpha(t)-1}I^*(i, t)$ and $C(1, t) = -\frac{\alpha'(i, t)\Lambda(i, t)}{\alpha(t)-1} + \left( \frac{\partial J^*(i, t)}{\partial s} \right) + \lambda^*(1, t)(p_0 - \frac{\alpha(t)}{\alpha(t)-1}J^*(1, t)) \geq \lambda^*(1, t)(p_0 - \frac{\alpha(t)}{\alpha(t)-1}J^*(1, t) - \frac{\alpha(t)}{\alpha(t)-1}J^*(1, t)) \geq 0$.

For $2 \leq i \leq n$, $C(i, t) = -\frac{\alpha'(i, t)}{\alpha(t)-1} + \left( \frac{\partial J^*(i, t)}{\partial s} \right) + \lambda^*(i, t)(I^*(i-1, t) - I^*(i, t)) \geq \frac{\alpha(t)-1}{\alpha(t)-1} \left( I^*(i-1, t) - I^*(i, t) \right) \geq \frac{\alpha(t)-2}{\alpha(t)-1} \left( I^*(i-1, t) - I^*(i, t) \right) \geq \frac{\alpha(t)-2}{\alpha(t)-1} \left( I^*(i-1, t) - I^*(i, t) \right)$.
1, t) − I∗(i, t)(I∗(i−1, t)1−α(t)−I∗(i, t)1−α(t))I∗(i−1, t)−I∗(i, t) − (1 − α(t))I∗(i, t)−α(t) ≥ 0, where the last inequality follows from the convexity of x1−α(s) (α(s) > 1). As the compensator is an increasing process, the claim follows immediately. □

Proof of Proposition 8.

Proof. By Lemma 6 and the optional sampling theorem of martingales (Karatzas and Shreve 1988).

□

Proof of Proposition 9.

Proof. By Dynkin’s formula, we only need to prove C(i, t) = ∂V ∂s (i, t) + j̸=i q(i, j)(V(j, t)−V(i, t)) = 0. It is easy to see that C(0, t) = 0. C(1, t) = −(α−1)1−αJ∗(1, t)−α ∂J∗(1, t) ∂s − λ∗(1, t)V(1, t) = (α − 1)2α−2α1−2αJ∗(1, t)1−2α(α − 1)2α−2α1−2αJ∗(1, t)1−2α = 0. For 2 ≤ i ≤ n, C(i, t) = −(α−1)1−αI∗(i, t)−α ∂J∗(i, t) ∂s + λ∗(i, t)(V(i − 1, t)−V(i, t)) = (α − 1)2α−2α1−2αI∗(i, t)−α(I∗(i, t)1−α − I∗(i − 1, t)1−α) + α1−2α(α − 1)2α−2I∗(i, t)−α(I∗(i − 1, t)1−α − I∗(i, t)1−α) = 0. Hence, V(N(s), s) is a F_s^{N}-martingale. The claim follows from the optional sampling theorem of martingales (Karatzas and Shreve 1988).

□

Proof of Proposition 10.

Proof. Let V(t; α) be a type independent value of time function. Then u(t; s) = V(t; α)−α(s)R(t; α, π), t ∈ [0, T]. By the first order condition, U′(s; α) = α(s)R′(s; α, π). It is easy to see that the claim holds after taking the integration.

□

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