Sequential Entry with Capacity, Price, and Leadtime Competition

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**Abstract**

In this paper we develop an analytic model to provide insight into strategic capacity planning in competitive environments. We consider two firms, an incumbent and a (potential) entrant, operating in make-to-order fashion and competing with identical products. Customers react to full price (the sum of nominal price and expected cost of delivery leadtime). The market is imperfect in the sense that firms can influence equilibrium full prices. We model firms as queuing systems, characterized by production variability—a proxy for operational efficiency—and capacity cost. Firms choose capacities (i.e., queuing service rates) according to a sequential game. The firms then engage in a simultaneous price game, with endogenously determined delivery leadtimes. In contrast to existing sequential entry models, this one includes a price (instead of quantity) post-entry subgame with a pure Nash equilibrium, incorporates “time to build,” and as a direct consequence of its assumptions, predicts the incumbent will never hold idle capacity to deter entry; this model predicts either a complete or a very strong first mover advantage. Depending on market conditions, this advantage can be overcome by an entrant whose operational or cost capabilities are vastly superior to those of the incumbent. The model also suggests that an entrant may gain from concealing information on his capabilities since that information can be very valuable in achieving market entry. However, depending on market conditions, the incumbent may be able to use capacity as a hedge against incomplete information.
1 Introduction

Capacity management has been well studied in the operations management field. Most traditional literature has focused on the tactical aspects of capacity, seeking to match productive capabilities to demand in the face of uncertainty (see Luss 1982 for a review). More modern capacity models have recognized that there is a link between capacity and responsiveness due to queueing behavior (Bitran and Tirupati 1989). But capacity management also contains a strategic dimension, since investments in capacity can play a strong role in defining a firm’s market position.

For example, consider entry in the Rambus DRAM (RDRAM) market. RDRAMs are high performance DRAM memory chips for PCs and other electronic products. In 1999 the top five DRAM suppliers controlled more than 80% of the global market (Harbert 2000) and provided standard chips to PC manufacturers, who placed a premium on price and delivery because rapid response times and low WIP levels had become significant profitability factors (Cullen 1999). An important determinant of responsiveness is capacity. However, DRAM manufacturers must typically add capacity in steps as complete facilities, which take one to two years to build and cost several billion dollars.

In 1997 Intel announced its plans to adopt RDRAMs for future PC memory requirements. Samsung, the long established DRAM leader, aggressively pursued leadership of the PC RDRAM market, becoming first to mass-produce the modules in August 1998. None of the other major DRAM makers challenged Samsung’s early lead, focusing instead on the evolutionary DDR technology, which required substantially lower initial capital expenditures. Although a vocal Rambus opponent, Micron—with its leading-edge production processes and manufacturing prowess—was expected by industry experts to enter the RDRAM market using a “fast follower” strategy. After the November 2000 launch of RDRAM-based Pentium 4 PCs, Samsung’s share of the global PC RDRAM market was around 80% (Robertson 2000). Toshiba and NEC shared the remaining 20% but were considered minor players who were merely extending their existing video game RDRAM operations to the PC market.

In an industry like this, where capacity is added in steps, production processes are highly variable, and customers are sensitive to both price and delivery leadtime, among the most pressing questions facing participants and analysts are: How strong is the incumbent’s advantage? Under what circumstances can an entrant be successful, and if so, how aggressive will the subsequent price competition be? How much can an entrant rely on operational efficiency to overcome the incumbent’s advantage?
As a first attempt to answer these questions, we develop a sequential game-theoretic model. We consider two firms: an incumbent and a (potential) entrant, operating in make-to-order fashion, and competing with a homogeneous product. Customers react to full price, defined as the sum of nominal price and expected cost of delivery leadtime. The market is imperfect in the sense that firms can influence equilibrium full prices. We model each firm as a single server queuing system, characterized by a production variability factor—a proxy for operational efficiency—and a capacity cost. Firms choose capacities (i.e., service rates) according to a sequential game. The incumbent sets her capacity first, which is observed by the entrant, and a nominal price for an initial monopoly period. The entrant then sets his own capacity, which is also observed by the incumbent, and the firms engage in a simultaneous price game for a second period, with delivery leadtimes determined endogenously. Thus, an equilibrium attains in which full prices are the same for both firms, and an arriving customer is indifferent among them. These equilibrium full prices depend on the individual characteristics of the firms.

To our knowledge, this is the first paper to use an analytic queueing model to study sequential entry with time competition. The model is also the first to incorporate a capacity game between firms with different operational or cost capabilities, and a subsequent price game where firms can influence equilibrium full prices.

We characterize the role capacity decisions play in a competitive market and how they are related to leadtime, firm capability, and customer preferences. If the firms are equally capable, depending on market conditions, the incumbent blockades, deters, or accommodates entry (always with higher profitability), so the first mover advantage is either complete or quite strong (both blockading and deterring prevent entry; see §5 for definitions). An entrant with superior operational or cost capabilities can overcome the advantage of the incumbent, and can even be the profit leader when the incumbent accommodates entry. However, this is unlikely since it requires vastly superior capability advantages on the part of the entrant. But the fact that the entrant may have better information about the incumbent than the incumbent has about the entrant may substantially increase the likelihood of successful entry. However, depending on market conditions and the accuracy of the incumbent’s forecast about the entrant, the incumbent can hedge against its lack of information through investment in capacity and still deter entry.

The rest of the paper is organized as follows. In §2 we relate the model to the literature. In §3 we formulate the model. In §4 we establish the types of possible behavior and describe the general solution. In §5 we evaluate the first mover advantage by analyzing the special
case of competition between an incumbent and an entrant which are otherwise identical. In §6 we analyze the possible outcomes of the game with heterogeneous firms and assess the possibility of overcoming the first mover advantage with improvements in operational efficiency and cost. We also evaluate the value of information on the entrant. The paper concludes in §7. All proofs are given in the Appendix.

2 Literature Survey

There is a vast literature on sequential entry in the industrial organization field of economics (work by Spence 1977 and Dixit 1980 is seminal; for reviews see Tirole 1988 and Allen et al. 2000.) However, existing capacity-constrained models ignore time competition. In addition, they cannot be used to analyze the strategic impact of a firm’s operational efficiency. Because capacity is treated as an abrupt constraint, these models must introduce explicit rationing rules when demand at certain prices exceeds total industry capacity, and different rationing rules lead to substantially different outcomes. For instance, assuming efficient rationing (a rule that maximizes consumer surplus), Kreps and Scheinkman (1983) showed that the outcome of the two-stage game considered by Levitan and Shubik (1972) in which the capacitated price game is preceded by a simultaneous capacity game, matches that of the single stage (Cournot) capacity game. However, Davidson and Deneckere (1986) found that the result does not hold for other rationing rules. With the exception of Allen et al. (2000), the literature also disregards capacity pre-commitments and assumes a quantity (Cournot) instead of price (Bertrand) post-entry competition. Assuming industry participants must pre-commit to fixed capacity levels sequentially is an attempt to capture the notion that capacity requires time to build.

Common criticisms of quantity competition are: (a) the need of a hypothetical auctioneer to set prices according to market quantities, and (b) the observation that in most situations prices are easier to adjust than quantities. One advantage of our model is that by including time competition, no rationing rules are needed. We show that as customers become insensitive to waiting or the system’s variance vanishes, the market clearing mechanism in our model coincides with the efficient rationing rule, thus supporting the results of Kreps and Scheinkman (1983) and the extensions of Osborne and Pitchik (1986) over the results of Davidson and Deneckere (1986). This model also incorporates sequential capacity pre-commitments and assumes a post-entry price competition. Furthermore, in contrast to Allen et al. (2000), whose post-entry solution is a mixed equilibrium, our post-entry game yields
a pure Nash equilibrium solution in a region of practical interest.

A controversial issue in this literature is whether an incumbent will hold idle capacity solely to deter entry. While Spence (1977) found that to be the case, Dixit (1980) showed that Spence’s solution depends on a non-credible threat (i.e., it was an imperfect equilibrium). Bulow et al. (1985) showed that Dixit’s conclusion depends on the assumption that each firm’s revenue is always decreasing in the other’s output and provided relevant cases where it is never satisfied, thus restoring Spence’s argument in a perfect equilibrium sense. Even if free, additional capacity can be detrimental because it prompts lower equilibrium prices and profits for both firms (Gelman and Salop 1985, Osborne and Pitchik 1986). A direct consequence of our modeling assumptions is that extra capacity can always be used to reduce expected leadtime and increase price without eroding equilibrium full prices, so contrary to Spence’s findings, firms never withhold capacity from the market.

Aside from being the first to use an analytic queueing model to study sequential entry with time competition, this paper extends the literature on pricing and capacity management in congestion systems. Models considering pricing in a single queueing system can be traced back to Naor (1969). Mendelson (1985) developed a model incorporating value with delay and capacity costs. Mendelson and Whang (1990) studied the design of pricing mechanisms to induce customers to behave according to a centralized system. Van Mieghem (2000) extended their results for dynamic scheduling and general delay cost functions. Other papers in this area include Dewan and Mendelson (1990), whose model was further analyzed by Stidham (1992) and by Rump and Stidham (1998).

Sobel (1973) appears to have been the first to analyze multiple queuing processes representing competing firms. He characterized the arrival process for a fixed population of homogeneous customers visiting one firm repeatedly and switching to another when an average of their historical delays exceeds an exogenous threshold. Li (1992) analyzed an oligopoly simultaneous game where firms compete on early delivery through inventory levers and found that competition can induce inventory holding. These papers assumed exogenous service rates, as well as exogenous and uniform prices.

Several authors have analyzed simultaneous price games between two queueing systems with exogenous service rates. Luski (1976), and Levhari and Luski (1978) considered competing identical firms when customers have heterogeneous costs of expected delay. They found that a pure Nash solution does not always exist. Li and Lee (1994) analyzed a two-firm model where upon arrival, homogeneous customers have perfect information on prices, service rates, and queue lengths, and join the queue that maximizes their utilities. Although
a pure Nash solution for their model does not always exist, they showed that when it does, the firm with a lower service rate offers a lower price. So (2000) studied price and time guarantees under competition assuming a market of fixed size where market share is determined by a multiplicative interaction model. In his model all firms use the same service level, effectively competing via a single degree of freedom. He characterized the Nash equilibrium for identical firms and performed a numerical analysis for heterogeneous firms.

All former price competition papers have assumed exponential service times. Lederer and Li (1997) developed a model where firms are differentiated by service time distribution and production costs. Customers are sensitive to full price and heterogeneous with respect to service requirements and cost of expected delay. They assumed fixed full prices, i.e., they cannot be individually influenced by any firm. Therefore, nominal prices and production rates are not independent, and firms compete by setting either price or rate for each customer type. They showed the existence of a unique equilibrium for the simultaneous game and characterized the solution. For homogeneous customers, a higher speed or a lower variability firm captures a larger market share and profit. A firm with higher production costs competes with faster delivery and higher prices. They analyzed important aspects of the solution when customers are heterogeneous. Although it is a reasonable approximation for markets with many firms and no entry, the fixed full price assumption is not suitable for modeling market entry in oligopolies. Lederer and Li (1997) recognized that “Removing the fixed full price assumption leads to several non trivial problems.” In our model we have embedded a simultaneous price game within a sequential capacity entry game. The post-entry game in our model contributes to the price game literature described here by assuming the firms—possibly with different service distributions—are capable of influencing full prices. Our price game model assumes customers have homogeneous costs of expected delay. However, customers are not completely homogeneous since individual reservation full prices may differ.

Kalai, Kamien, and Rubinovitch (1992) were first to analyze a simultaneous service rate game between two queuing systems. In their model, Poisson service arrivals join a common queue shared by two exponential servers with identical costs and quality of service. From the head of the queue, requests proceed either to the first available server or are randomly routed when both servers are available. The authors showed that a symmetric stable equilibrium exists when costs of service are low relative to rewards. Gilbert and Weng (1998) extended Kalai et al. (1992) to study competition between two servers and a coordinating agency which minimizes its costs while keeping cycle time lower than an exogenous bound. The servers select their capacities to maximize profits while the agency sets the payment per unit
service and decides whether to use a single or separate queues. They found conditions for the agency to select separate queues. That arrangement fosters more intense competition, with the servers installing relatively higher levels of capacity, which offsets the pooling benefits of a single queue. Hall and Porteus (2000) focused on customer loyalty over multiple periods. In their model customers are not rational decision-makers but instead may react to denials of service by switching firms, so our results and theirs are not directly comparable. These three papers consider simultaneous service rate competition and, like Gans (2002), who analyzed customer loyalty in a game between suppliers competing on quality levels, all assume fixed and identical prices and a fixed market size with no entry.

To date, only Reitman (1991), Loch (1992), and Cachon and Harker (2002) have developed models with delay sensitive customers and firms competing both on capacity and price. Instead of using a queueing model, Reitman assumed waiting time increases linearly in utilization. He focused on short-term capacity decisions and analyzed an oligopoly game where identical firms simultaneously choose capacity and price, with sales volume determined by the market. He found that when customers have heterogeneous costs of delay, the solution is symmetric for a duopoly and asymmetric for three or more firms. Cachon and Harker considered a simultaneous price and capacity game between two M/M/1 queues indirectly by analyzing a game where firms, which are also allowed to outsource production, choose full prices. Loch formulated a two-stage game between two queuing systems, in which firms initially compete on service rates simultaneously and subsequently on quantities, with prices determined by the market (i.e., Cournot competition). His numerical analysis of exponential servers suggested that multiple equilibria can occur, with either a single firm with positive capacity or two identical firms. Our model is most similar to Loch’s, with the main difference being that we consider capacity competition as a sequential game followed by a price game with quantities determined by the market (i.e., Bertrand competition). Although technically more challenging, we believe Bertrand competition is more realistic in most settings.

3 Model Formulation

We consider two firms, an incumbent (“she”) and a potential entrant (“he”) that offer identical products in a market where customers are sensitive to price and delivery leadtime. The incumbent makes her production capacity decision first, and acts as a price-setting monopolist for an initial period of time. The entrant then makes his capacity decision, the firms engage in price competition and the system reaches equilibrium for a second time.
period. We first give a basic model formulation, and establish the market clearing conditions, which determine the equilibrium supply rate of each firm for any given set of capacities and prices. We then proceed backwards in time, first analyzing price decisions for a given choice of capacities, and then the sequential capacity game.

3.1 General Description

We can describe the model in terms of four stages and two periods, with each firm making a single capacity decision. In the first stage, firm 1—the incumbent—installs her capacity or production rate $\mu_1$. She also sets her price $p_0$ for the first time period, in which she behaves as a profit-maximizing monopolist. In the second stage, after observing $\mu_1$, firm 2—the entrant—installs his capacity $\mu_2$, which firm 1 observes. Once installed, firms cannot sell or trade any portion of their capacity. In the third stage firms set prices for the second time period according to a profit-maximizing simultaneous subgame, with firm $i$ setting price $p_i$, for $i = 1, 2$. In the fourth stage, demand is allocated according to the market clearing conditions (which we describe in §3.2), and firms realize second period profits. Notice that we use subscripts to label players. We will use superscripts for constants related to parameters and a tilde to denote unit transformations.

We assume that market demand-rate functions, $D_0(\cdot)$ and $D(\cdot)$, for the first and second periods, respectively, depend only upon full price $\pi = p + aw$, where $p$ is nominal unit price, $w$ is expected leadtime, and $a$ is cost of delay per unit time. We also assume there exists a finite $\pi_M$ such that both functions are continuous and strictly decreasing with support $[0, \pi_M]$. Notice that although $a$ is identical for all customers, each may have a distinct reservation full price, defined as the maximum full price a customer is willing to accept and still purchase the product. Customers cannot resell the products to each other. For simplicity $a$ is period independent, but none of our results depend on this assumption.

The expected leadtime for firm $i$ is

$$w_i = \frac{v_i}{\mu_i} \frac{\lambda_i}{\mu_i - \lambda_i} + t_0, \quad i = 1, 2,$$

(1)

where $\lambda_i$ is its supply rate, $v_i$ is a dimensionless variability parameter, and $t_0$ is the raw process time (i.e., without queueing delays). When $v_i = (c_a^2 + c_s^2)/2$, with $c_a^2$ and $c_s^2$ the squared coefficients of variation of the inter-arrival and service times respectively, the first term in (1) is a heavy-traffic approximation for the expected steady-state delay of the GI/G/1 queue, which is exact for the M/G/1 (Whitt 1993). Hence, we ignore any transient queueing behavior at the beginning of each period. Notice that we assume $t_0$ is independent of $\mu_i$, 

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which is not characteristic of a GI/G/1 queue. However, this modeling assumption attempts to reflect the fact that a typical production facility resembles one or more tandem lines of queues with parallel servers, where adding capacity to the bottleneck affects queuing but has minimal effect on raw process time. Hence, we represent the leadtime as a GI/G/1 queue followed by a fixed raw process time.

3.2 Market Clearing Conditions

In the first period, when firm 1 behaves as a monopolist, to determine demand allocation uniquely it suffices to assume market demand matches supply, i.e., \( D_0(p_0 + a w_0) = \lambda_0 \), where \( w_0 \) and \( \lambda_0 \) denote the expected monopoly leadtime and supply rate, respectively. In the second period, for any capacity and price choices \( \mu_i \) and \( p_i, i = 1, 2, \) the market clearing conditions:

\[
p_1 + a w_1 = p_2 + a w_2, \quad (2)
\]

\[
D(p_1 + a w_1) = \lambda_1 + \lambda_2. \quad (3)
\]

determine demand allocation, namely \( \lambda_1 \) and \( \lambda_2 \). Equation (2) states that in equilibrium, customers are indifferent among firms, and (3) matches aggregate market demand rate at the equilibrium full price with the combined supply rate of the firms. A unique solution \((\lambda_1, \lambda_2)\) for these conditions always exists (see the Internet Appendix). Moreover, \( \lambda_i > 0, i = 1, 2 \) as long as neither firm prices herself out of the market (i.e., \( p_m + a t_0 < \pi_M \) and \( p_m \leq \hat{p} \)), where \( m = \arg \min \{p_i : i = 1, 2\} \), \( \overline{m} = \arg \max \{p_i : i = 1, 2\} \), \( \hat{p} = p_m + a (w_m(\gamma) - t_0) \), and \( \gamma \), the solution to \( D(p_m + a w(\gamma)) = \gamma \), denotes the monopoly supply rate of firm \( m \). Notice that \( D(\cdot) \) being strictly decreasing ensures stable queuing behavior (i.e., \( \lambda_i < \mu_i \) for \( i = 1, 2 \)). A useful property of the solution is that increasing the nominal price of firm \( i \) while holding everything else constant (except the supply rates) causes an increase in the equilibrium full price, a decrease in the combined supply rate, and a decrease (increase) in supply rate of firm \( i \) \((j)\), where \( j \neq i \). Analogously, increasing the capacity of firm \( i \) while holding everything else constant (except the supply rates) causes an increase in the combined supply rate, a decrease in the equilibrium full price, and an increase (decrease) in the supply rate of firm \( i \) \((j)\). See the Internet Appendix for proofs.

Henceforth, we assume the explicit demand rate function

\[
D(\pi) = (\Lambda - b \pi)^+ \quad (4)
\]

where \( x^+ = \max(x, 0) \) and \( \Lambda \) and \( b \) are constants. This demand results from assuming customer’s reservation full prices are uniformly distributed on \([0, \Lambda/b]\), since \( D(\pi) = \int_0^{\Lambda/b} b du = \)
Choosing proper units for time and money, without loss of generality we set $b = \Lambda - a b \tilde{t}_0 = 1$ (the required unit transformation is given in Proposition 1 below). For convenience we denote a quantity in arbitrary units by $\bar{x}$ and the equivalent quantity in normalized units by $x$. With this and (4) the market clearing conditions become

$$p_i + \frac{\alpha v_i}{\mu_i} \frac{\lambda_i}{\mu_i - \lambda_i} = 1 - \lambda_1 - \lambda_2, \quad i = 1, 2. \quad (5)$$

**Proposition 1.** If $\tilde{\Lambda} > \tilde{a} \tilde{b} \tilde{t}_0$, setting $b = \Lambda - a b \tilde{t}_0 = 1$ implies $a = \tilde{a} \tilde{b} (\Lambda - \tilde{a} \tilde{b} \tilde{t}_0)^{-2}$. Prices and capacities scale according to $\tilde{p} = p (\tilde{\Lambda} - \tilde{a} \tilde{b} \tilde{t}_0) / \tilde{b}$ and $\tilde{\mu} = \mu (\Lambda - \tilde{a} \tilde{b} \tilde{t}_0)$.

We assume $\tilde{D}(\tilde{a} \tilde{t}_0) = \tilde{\Lambda} - \tilde{a} \tilde{b} \tilde{t}_0 > 0$ to avoid trivialities. In addition to relating the original and normalized solutions, Proposition 1 provides information about the sensitivity of the equilibrium actions $\tilde{\mu}_i, \tilde{p}_i, i = 1, 2$ to changes in the parameters $\tilde{\Lambda}, \tilde{a}, \tilde{b}$, and $\tilde{t}_0$. Specifically, we can conclude: (1) as the maximum demand rate increases, while keeping everything else constant, equilibrium capacities and prices increase; (2) equilibrium capacities and prices decrease when the market is more sensitive to changes in full price or raw process time $\tilde{t}_0$ increases.

The market clearing conditions assume the presence of system variability and delay-sensitive customers. Absent either of these, the assumptions of traditional models obtain. To compare this model with the capacity-constrained price competition literature, we analyze the limiting behavior of the market clearing conditions.

**Proposition 2.** Let $p_i \leq p_j, \mu_i > 0$. As $a \to 0$ or $v_i \to 0$,

1. If $p_j \geq \max\{p_i, 1 - \mu_i\}$ then $\lambda_i \to \min\{\mu_i, 1 - p_i\}$, $\lambda_j \to 0$
2. If $p_i < p_j < 1 - \mu_i$ then $\lambda_i \to \mu_i, \lambda_j \to \min\{1 - p_j - \mu_j, \mu_j\}$
3. If $p_i = p_j = p < 1 - \mu_i$ then
   a) $\lambda_i \to \mu_i, \lambda_j \to \mu_j$ if $\mu_i + \mu_j \leq 1 - p$,
   b) $\lambda_i \to \gamma_i, \lambda_j \to \gamma_j$ if $\mu_j < 1 - p < \mu_i + \mu_j$,
   c) $\lambda_i \to 0, \lambda_j \to 1 - p$ if $1 - p \leq \mu_j$.

where $\gamma_i = \alpha - \sqrt{\alpha^2 - 4\beta}/2(v_j/\mu_j - v_i/\mu_i)$, $\lambda_j = (v_i \mu_j \gamma_i/\mu_i)/[(\mu_i - \gamma_i) v_j/\mu_j + v_i \gamma_i/\mu_i]$ if $v_j/\mu_j > v_i/\mu_i$; $\gamma_i = [(1 - p) \mu_i]/[\mu_i + \mu_j]$, $\lambda_j = [(1 - p) \mu_j]/[\mu_i + \mu_j]$ if $v_j/\mu_j = v_i/\mu_i$, with $\alpha = (1 - p) (v_j/\mu_j - v_i/\mu_i) + v_j \mu_i/\mu_j + v_i \mu_j/\mu_i$, $\beta = (1 - p) (v_j/\mu_j - v_i/\mu_i) v_j \mu_i/\mu_j$. ($v_j/\mu_j \geq v_i/\mu_i$ without loss of generality in (b)).

This demand allocation coincides with the efficient rationing rule, which includes a discontinuity when firms match prices as in (3). The outcome of the price game is not affected
by the specific allocation at the discontinuity (e.g., Kreps and Scheinkman 1983 allocate
demand equally between firms, while Osborne and Pitchik 1986 do it in proportion to ca-
pacities). Therefore, as discussed in §2, this model supports the efficient rationing rule and
ensuing results, e.g., Kreps and Scheinkman’s (1983) and Osborne and Pitchik’s (1966), over
those of Davidson and Deneckere (1986).

3.3 Simultaneous Price Subgame

To begin, we assume the firms have made their capacity decisions and focus on the price
subgame. Hence, let \( v_i \) and \( \mu_i \) for \( i = 1, 2 \) be arbitrary, positive, and common knowledge. We
assume zero marginal production costs, but the model also captures the case of non-zero and
identical marginal production costs since \( p_i \) is the unit margin. We define the simultaneous
price subgame via the reaction price functions. Because at this stage capacity costs have
been incurred, firms set prices to maximize revenues. Given \( p_j, j \neq i \), we define firm \( i \)'s
reaction price \( p_i \) according to

\[
\arg \max_{p_i \geq 0} p_i \lambda_i(p_i, p_j)
\]

where \( \lambda_i(p_i, p_j) \) is the solution to (5) when \( p_i \leq p^c(p_j; a v_j/\mu_j, \mu_j) \) and zero otherwise, with
\( p^c(p; y, \mu) = (1 + p - y - \mu + [(1 - p + y + \mu)^2 - 4(1 - p)\mu]^{1/2})/2 \) representing the critical
price beyond which firm \( i \) prices herself out of the market. Note that the model implicitly
assumes that both firms make all their installed capacity available to the market. However,
this is without loss of generality, because restricting the use of any portion of capacity is sub-
optimal since capacity costs are sunk and customers are indifferent among all combinations
of nominal price and expected delay resulting in the same full price. Therefore, a firm can
always use any extra capacity to reduce expected delays and increase her nominal price so
as to keep equilibrium full prices unaffected. This leaves captured demand for both firms
unchanged, and results in higher revenues for the firm with extra capacity by virtue of the
price increase. See the Internet Appendix for a formal proof.

A pure-strategy solution does not always exist for subgame (6). An example is the
instance with \( a = .5, v_1 = 0.004, v_2 = 0.8, \mu_1 = 0.6, \) and \( \mu_2 = 1.8 \) (see the Internet
Appendix for an illustration and expanded discussion). It is not surprising that this occurs
for low variabilities, since the same is true in traditional models when capacities are relatively
large (see Levitan and Shubik 1972). In the following proposition we define a region of
practical interest in which a pure-strategy solution to the price subgame always exists. This
is significant because mixed strategies are unsettling since an equilibrium price may be
optimal ex ante but not ex post, tempting firms to change prices then.

**Proposition 3.** For every $\eta > 0$, there exists $\overline{p}(\eta) > 0$ such that a pure-strategy solution for the Nash price subgame exists if $a v_i \geq \eta$ and $\mu_i \leq \overline{p}(\eta)$ for $i = 1, 2$. In particular, $\overline{p}(0.05) = 1.32$, $\overline{p}(0.1) = 1.95$, $\overline{p}(0.25) = 5.20$ and $\overline{p}(0.5) = 24.5$.

Whether $\mu_i \leq \overline{p}(\eta)$ is of consequence in practice depends on capacity costs. To construct an indirect measure let $c^m(\eta)$ be the cost that makes $\overline{p}(\eta)$ optimal for the monopolist, and $\varphi(\eta) = (p^* \lambda^*/c^m(\eta)\overline{p}(\eta) - 1)$ her “return on investment.” Straightforward calculations (discussed in §4.1) yield $\varphi(0.05) = 8.34$, $\varphi(0.1) = 11.9$, $\varphi(0.2) = 28.8$, so $\mu_i \leq \overline{p}(\eta)$ should be of no practical consequence for $\eta \geq .05$. Proposition 3 may be used with caution for $\eta \in [0.0075, 0.01]$ ($\varphi(\eta) \in [1.87, 2.16]$), but not for $\eta \leq .001$ ($\varphi(.001) = 0.16$). We explore this further in §5 and §6. Given $\eta_i$, $i = 1, 2$, the regions of $\mu_i$ in which a solution to (6) may not exist are bounded from below and above, and $\overline{p}$ are only lower bounds for those regions. Indeed, in §4 we analyze a game in which one of the capacities is infinite and the solution to the price subgame is still well-defined. Because multiple solutions may give rise to an unstable or ambiguous equilibrium, it is desirable to establish that the solution to (6) is unique. Unfortunately, we have been unable to do so. Therefore, all our numerical calculations incorporate strategy space searches to ensure a unique price subgame solution and avoid any misrepresentation. None of our extensive numerical results have provided an example of multiple solutions to (6).

### 3.4 Sequential Capacity Game

Having established pricing and demand allocation procedures, we now address capacity decisions. Although it is not uncommon for production capacity costs to decline in time (e.g., because of technological advances), we focus on the initial stages of the product life cycle when capacity costs changes are not significant. Let $\tau_k$ denote the duration of period $k$ for $k = 1, 2$, $c_i$ the cost per unit production rate, and $c_i = \hat{c}_i/\tau_2$ for $i = 1, 2$. Given $a$, and $v_i, c_i$, $i = 1, 2$, the firms make their capacity decisions to maximize their profits according to

$$
\mu^*_2(\mu_1) = \min \left\{ \arg \max_{\mu_2 \geq 0} \left\{ p^*_2(\mu_1, \mu_2) \lambda^*_2(\mu_1, \mu_2) - c_2 \mu_2 \right\} \right\}, \quad \text{for } \mu_1 \geq 0,
$$

$$
\mu^*_1 = \min \left\{ \arg \max_{\mu_1 \geq 0} \left\{ p^*_0(\mu_1) \lambda^*_0(\mu_1) \tau + p^*_1(\mu_1, \mu_2^*(\mu_1)) \lambda^*_1(\mu_1, \mu_2^*(\mu_1)) - c_1 \mu_1 \right\} \right\},
$$

where $\tau = \tau_1/\tau_2$, $p^*_0(\mu_1)$ and $\lambda^*_0(\mu_1)$ are the optimal monopoly price and supply, and $p^*_1(\mu_1, \mu_2)$ and $\lambda^*_i(\mu_1, \mu_2)$ for $i = 1, 2$, are the equilibrium values in the price subgame’s solution from
§3.3. Sequential game (7) is well-defined because profits are continuous and bounded from above for $\mu_i \geq 0$, diverging to $-\infty$ as $\mu_i \to \infty$ for $i = 1, 2$. The minimum function resolves ambiguities should multiple maxima arise.

Notice that the incumbent, firm 1, must commit her capacity for the game’s two periods with a single decision, whereas the entrant commits only for the second period. The first period generates supplemental revenue for the incumbent, and hence serves as a first mover advantage. The magnitude of the first period revenue depends mainly on the first period demand. However, since our main goal is to analyze incumbent behavior independently of any revenues from the initial monopoly phase, we henceforth assume $D_0(\cdot) \equiv 0$.

4 Bounds on Competitive Behavior

To characterize the model’s general solution we analyze two extreme cases: (1) a monopoly, which represents the best possible case for the incumbent, and (2) a market with an infinite capacity incumbent, which represents the worst possible case for the entrant. For comparison purposes, we also consider the case where nominal prices are fixed exogenously.

4.1 Monopoly

Suppose there is no entrant. Then, for a fixed capacity, equations (5) reduce to $(\lambda - \mu)(\lambda + p - 1) \mu - av \lambda = 0$, whose only root satisfying $\lambda < \mu$ is $\lambda(p) := (1 - p + av/\mu + \mu - [(1 - p + av/\mu + \mu)^2 - 4 \mu (1 - p)]^{1/2})/2$. We omit the subscript throughout this subsection since there is only one firm. We choose the optimal pricing strategy $p^* = p^*(\mu)$ to maximize revenue $p \lambda(p)$, which is continuous and concave (see Internet Appendix for details). Hence $p^*(\mu)$ is unique, and given $\mu$, can be computed using standard numerical techniques.

The capacity decision involves maximizing the profit function $p^*(\mu) \lambda(p^*(\mu)) - c \mu$. During a numerical evaluation of the first term—varying $\mu$ for an extensive set of values of $\eta = av$ of practical interest—we found no cases where it was not convex-concave (i.e., convex up to an inflection point and concave thereafter), so we assumed it was convex-concave in general. Therefore, the capacity derivative of revenue is increasing in $\mu$ up to some $\mu^h$, where it reaches its maximum, and decreasing thereafter. Hence, when $c \geq c^h := d(p^*(\mu^h) \lambda(p^*(\mu^h))) / d\mu$, profit is decreasing in $\mu$ for every $\mu \geq 0$, so $\mu^* = 0$. When $c < c^h$, the first order condition $d(p^*(\mu) \lambda(p^*(\mu))) / d\mu = c$ has two solutions, corresponding to a local minimum and maximum. Here, the optimum capacity is the greater of the local maximum
and zero (the local maximum need not be positive). In this manner, the optimum price policy and capacity can be computed for any \( c \) and \( \eta = a \cdot v \) of practical interest.

Generating a plot of isoprofit curves in \((c, \eta)\) we quantified, in the region of interest, the value of variability reductions to a monopolist. The zero isoprofit, whose parametric form we denote by \( c^u(\eta) \), determines the market region \( \{(\eta, c) : 0 \leq \eta, 0 \leq c \leq c^u(\eta)\} \). Outside this region, any capacity investment is unprofitable for a monopolist, and therefore for each of the firms in a duopoly as well. Iso-capacity plots show that capacity decisions decrease with \( c \) when \( \eta \) is held constant, and increase with \( \eta \) when \( c \) is held constant. Iso-price plots reveal that price moves opposite to capacity, increasing (decreasing) with \( c \) (\( \eta \)) when \( \eta \) (\( c \)) is held constant. In addition, relative price variation is substantially lower than relative capacity variation (for \( 0 \leq \eta \leq 1 \), \( \mu^* \) varies roughly between 0.75 and 3, whereas \( p^* \) remains roughly between 0.5 and 0.53 throughout the market region).

### 4.2 Infinite Capacity Incumbent

To draw inferences about the structure of the general solution, we analyze the extreme case of an entrant competing with an infinite capacity incumbent, i.e., \( c_1 = 0 \). Here, the market clearing conditions become \( p_1 = p_2 + v_2 \lambda_2/\mu_2 \) \((\mu_2 - \lambda_2)\), \( p_1 = 1 - \lambda_1 - \lambda_2 \), solved by \( \lambda_1 = 1 - p_1 - (p_1 - p_2) \mu_2/(p_1 - p_2 + v_2/\mu_2) \), \( \lambda_2 = (p_1 - p_2) \mu_2/(p_1 - p_2 + v_2/\mu_2) \). Given \( \mu_2 > 0 \), the reaction functions of the simultaneous price subgame are defined by \( \max\{u_2(p_2) := p_2 \lambda_2(p_1, p_2) : 0 \leq p_2 \leq p_1\} \) for the entrant (firm 2), and \( \max\{u_1(p_1) := p_1 \lambda_1(p_1, p_2) : p_2 \leq p_1 \leq p_1 \leq p_1 \leq p_1 \) \( p_1 > p_2 \) \( v_2/\mu_2, \mu_2) \} \) for the incumbent (firm 1; see §3.3). Since \( u''_2(p_2) = -2v_2(p_1 + v_2/\mu_2)/(p_1 - p_2 + v_2/\mu_2)^3 < 0 \) for \( p_2 \leq p_1 \), so \( u_2(\cdot) \) is concave, the entrant’s reaction price is the unique solution to the first order condition \( u'_2(p_2) = 0 \), or \( p^*_2(p_1) = p_1 + v_2/\mu_2 - [(p_1 + v_2/\mu_2)^2/\mu_2]^{1/2} \). In contrast, \( u_1(\cdot) \) is not concave. In the proof of Proposition 3 we showed that revenue is pseudoconcave in a region of practical interest for finite \( \mu_1 \). Proposition 4 establishes the analogous region of practical interest for the case where the incumbent has infinite capacity.

**Proposition 4.** If the incumbent has infinite capacity, sufficient conditions for the existence of a pure strategy solution to the simultaneous price subgame are a \( v_2 \geq .25 \), or \( \mu_2 \geq 1 \). Otherwise, given rational \( \eta_2 = a \cdot v_2 < .25 \) and \( \mu_2 < 1 \), existence can be established analytically by root isolation of a cubic polynomial. If \( (p^*_1, p^*_2) \) is a pure strategy solution, \( p^*_1 > p^*_2 > 0 \).

As they did for the finite capacity game, our numerical calculations incorporate searching the strategy space to rule out multiple solutions to the price subgame. Although the entrant’s
price is lower than the infinite capacity incumbent’s, it is noteworthy that it is not zero. This implies, as stated in the next proposition, that as long as its capacity cost is sufficiently low, an entrant facing an infinite capacity incumbent will be profitable.

**Proposition 5.** If the incumbent has infinite capacity and the price subgame has a pure strategy solution, for every \( \eta_2 = a v_2 \) there exists \( c^\infty(\eta_2) > 0 \) such that the entrant is profitable if and only if \( 0 \leq c_2 < c^\infty(\eta_2) \).

Because the entrant’s profitability can only increase when the incumbent’s capacity decreases, the previous result clearly extends to the general game (i.e., \( 0 < c_i < \infty \) and \( \eta_i = a v_i > 0, i = 1, 2 \)). Therefore, there exists \( c_i^f(c_j) = c_i^f(c_j, \eta_i, \eta_j) > 0 \) with \( c^\infty(\eta_i) < c_i^f(c_j, \eta_i, \eta_j) < c^u(\eta_j), j \neq i \) such that firm \( i \) is profitable if and only if \( c_i < c_i^f(c_j) \). We may thus use the monopoly and this subsection’s extreme case to describe the structure of the general solution. Given \( (c_j, \eta_i, \eta_j) \) there exist \( c^u(\eta_j) \geq c^d(c_j, \eta_i, \eta_j) \geq 0 \) such that if \( c_i \geq c^u(\eta_j) \) neither firm participates in the market, if \( c_i \in [c^d(c_j, \eta_i, \eta_j), c^u(\eta_j)] \) only firm \( j \) participates, and if \( c_i \in (0, c^d(c_j, \eta_i, \eta_j)) \) both firms participate. Within each region, profitabilities continuously deteriorate as capacity costs increase.

### 4.3 Fixed Prices

For comparison purposes, it is instructive to consider the special case where prices are fixed exogenously, with \( p_1 = p_2 = p \). Although uncommon, this case applies in some settings. An example is competition among local vehicle road-side assistance service providers holding contracts with national companies (e.g., AAA). The national company sets fees (i.e., prices), and routes service requests from its subscribers to local contracted providers according to shortest expected waiting time.

Here, the profit functions in (7) are proportional to \( r_2(\mu_2; \mu_1) = \lambda_2(\mu_2, \mu_1) - c_2 \mu_2 \) and \( r_1(\mu_1; \mu_2) = \lambda_1(\mu_1) \tau + \lambda_1(\mu_1, \mu_2) - c_1 \mu_1 \), with \( \lambda(\cdot) \) the optimal monopoly fixed-price supply (throughout this subsection, for convenience we replace the condition \( \Lambda - a b t_0 = 1 \) with \( \Lambda - a b t_0 - b p = 1 \) and redefine \( c_i = \hat{c}_i/(p \tau_2) \) for \( i = 1, 2 \)). Although the capacity reaction functions can have multiple discontinuities, they are well behaved in a region of practical interest (for \( i = 1, 2, a v_i \geq 0.002 \) suffices to ensure \( \lambda_i(\cdot; \mu_j) \) is convex-concave for every \( \mu_j \geq 0 \), where \( j \neq i \); see Internet Appendix for details). In the following proposition we characterize the fixed-price, sequential capacity game solution when the entrant does not have a cost or variability advantage.
Proposition 6. For fixed identical prices and any \( D_0(\cdot) \geq 0 \), if \( v_1 \leq v_2 \) and \( c_1 \leq c_2 \), the incumbent captures the entire market.

In particular, the above result holds when there is no initial demand (\( D_0(\cdot) \equiv 0 \)) and no cost or variability advantage on the part of the incumbent (\( c_1 = c_2 \) and \( v_1 = v_2 \)). This stands in sharp contrast to the simultaneous pure-strategy solution under the same conditions, which, when it exists, must be symmetric (this is analogous to the solution of Kalai et al. 1992 but for a different model). Hence, the incumbent attains total market dominance through capacity pre-commitment and queueing economies of scale. However, as the extension of Proposition 5 illustrates, price competition erodes the incumbent’s power, since it may force her to accommodate entry.

5 First Mover Advantage

Although the entrant can be profitable, being first clearly confers an advantage on the incumbent. To quantify this advantage, in this section we analyze equally capable firms with \( v_i = v \) and \( c_i = c \) for \( i = 1, 2 \). Our numerical examples (in this and subsequent sections) consider parameters that cover the entire range of the region of practical interest. To estimate the order of magnitude of \( a = \tilde{a} \tilde{b}[\tilde{\Lambda} - \tilde{a} \tilde{b} \tilde{t}_0]^{-2} \) we use the following DRAM industry data: 1998 had periods of oversupply and sharp price drops, so \( \tilde{\Lambda} - \tilde{a} \tilde{b} \tilde{t}_0 \approx \tilde{\lambda} \) and \( a \approx \tilde{a} B/(\tilde{\lambda} \tilde{p}) \), with \( -B \) the elasticity of demand. The average computer price change that year was \(-35\%\), with an average price of $1,949 (Aizcorbe et al. 2000). Memory content was 63.3MB per computer (IDC), at an average of $1 per MB. Using a 15% “hurdle” rate (www.sematech.org), \( \tilde{a} \approx ((.15 + .35)/\text{year})(\$1,949/\$63.3)\tilde{p} = 15.35 \tilde{p}/\text{year} \). On average \( B = 2.7 \) (Zulehner 2003) and \( \tilde{\lambda} \approx 96 \) based on monthly orders from the 8 major OEM’s, so \( a \approx (15.35)(2.7)/96 = 0.43 \). Using \( v = .75 \) for this industry results in \( \eta = a v \approx .32 \). Hence, we choose \( \eta = .5 \) as a typical value.

Figure 1 depicts the solution for \( \eta = 0.5 \) as a function of \( c = \tilde{c} \tilde{b}/(\tilde{\Lambda} - \tilde{a} \tilde{b} \tilde{t}_0) \). When \( c \) is too high, i.e., \( c \geq c^a \), no firm is profitable. When \( c^b \leq c < c^a \), the market is small and can only bear one firm. Here, the incumbent’s decisions and profits are no different from a monopolist’s, and entry is said to be blockaded.

As \( c \) decreases to \( c^d \leq c < c^b \), the incumbent is still able to prevent entry, but not without installing additional capacity beyond the monopoly level, so entry is deterred. The additional capacity lowers the equilibrium full price \( \pi \) to the point where the capacity investment required by any potential entrant to achieve a sufficiently short leadtime \( w < \pi \), cannot be
covered with revenues from the small price $\pi - w$ it can charge. In traditional models (e.g., Allen et al. 2000), the incumbent often installs capacity ex ante to deter entry but does not use all of it ex post when optimizing revenues, with idle capacity serving only as a “credible threat.” As we discussed in §3.3, time competition in our model leads the incumbent to use all available capacity ex post, which in our view is more realistic in most settings.

Figure 1 depicts an interesting feature of the solution: When deterring entry, although the incumbent acts to reduce full price’s controllable part ($\pi - at_0$), it virtually matches monopoly prices. Hence, in contrast to traditional models, the threat of entry actually increases the incumbent’s revenues (although more slowly than it increases capacity costs), and benefits customers by reducing leadtimes and increasing consumer surplus.

Notice in Figure 1 how as $c$ approaches $c^d$ from above, $p_1$ becomes closer to $\pi - at_0$, so expected delays $(\pi - p_1 - at_0)/a$ become smaller. Also, capacity costs increase rapidly until, as discussed in §4.2, the incumbent can no longer deter entry, and therefore accommodates it when $c < c^d$. In that case the incumbent installs less capacity, but still more than the monopolist and substantially more than the entrant. Firms specialize, with the entrant offering lower prices and longer leadtimes. Nominal prices are not only lower than in the no entry regions, but also drop faster as $c$ decreases because of post-entry competition. Figure 1 also shows that with accommodated entry, customers enjoy lower full and nominal prices, but not always shorter delays than with deterred entry.

Analysis of the solution for additional values of $\eta$ in the range $[0.25, 1]$ revealed the
same qualitative features. As $\eta$ grows, region boundaries ($c^d, c^b, c^u$) decrease, becoming (.049, .085, .198), (.035, .061, .149), and (.025, .043, .111) for $\eta = .25, .5$ and 1 respectively. Clearly, the first mover advantage is absolute when $c \geq c^d$. To quantify the magnitude of the advantage when $c < c^d$ and entry is accommodated, we calculate the ratio of optimal profits, which we plot as a function of $c$ in Figure 2. Notice that when $0.25 \leq \eta \leq 1$, $0.07 \leq \Pi_2/\Pi_1 \leq 0.21$ so the advantage, though partial, is strong. Since the entrant’s capacity is lower, the advantage seems less prominent from a return on investment point of view (i.e., $(\Pi_2/\mu_2)/(\Pi_1/\mu_1)$ ranges between 0.25 and 0.97), but the meaning of that comparison is ambiguous in light of the fact that firms maximize profits instead of return on investment.

![Figure 2: Relative profits of equally capable firms ($\eta_1 = \eta_2 = \eta, c_1 = c_2 = c$) when entry is accommodated.](image)

6 Overcoming the First Mover Advantage

Earlier, we focused on equally capable firms to determine the nature of the first mover advantage. To find how an entrant may overcome that advantage, we now study an entrant with superior capabilities; specifically: variability, capacity cost, and information advantages.

6.1 Variability Advantage

To study firms with different operational efficiencies and identical capacity costs, we set $\eta_1 = a v_1 = 0.5$ (as in §5), $c_1 = c_2 = c$, and solve the model numerically in the $(c, \eta_2)$ area of practical interest. By making use of an adaptive sampling strategy that computed the solution for extra points near non-trivial boundaries and applying Mathematica’s interpolation algorithms where needed, we generated the optimal policy structure shown in Figure 3. The solid curves define four main regions; three of which are divided into subregions by the dashed curves. The horizontal dotted line at $\eta_2 = a v_2 = 0.5$ corresponds to the instance depicted in Figure 1. Recall that when $c \geq c^u(\eta_1)$, the incumbent is unprofitable even as a monopolist, and analogously for the entrant if $c \geq c^u(\eta_2)$; this is the unprofitable region.
Figure 3: Overcoming the first mover advantage with variability \((av_1 = 0.5, c_1 = c_2 = c)\).

The **incumbent only** region includes the **blockaded entry** and **deterred entry** subregions. For any fixed \(c < c^u(\eta_1)\), a \(\eta_1\) monopoly is profitable. If \(\eta_2\) is sufficiently high, the incumbent can make entry unprofitable just by installing the capacity of a \((c, \eta_1)\) monopolist; this defines the **blockaded entry** subregion. As \(\eta_2\) decreases into the **deterred entry** subregion, the incumbent must install capacity beyond the monopoly level to make entry unprofitable. As we discussed in §5, in this case the incumbent uses all available capacity ex post.

The **entrant only** region consists of the **vanquished incumbent** and **defeated incumbent** subregions. When \(\eta_2 < \eta_1\), the cost may be too high for a monopoly with a parameter \(\eta_1\) but not for one with \(\eta_2\), hence, the **vanquished incumbent** subregion, defined by \(c^u(\eta_1) \leq c < c^u(\eta_2)\), defines the situation where the entrant’s decisions and profits match those of a monopolist’s, representing the reverse of a blockaded entry scenario. In this case the incumbent’s absence and entrant’s presence result only from their capabilities, regardless of competition. When \(\eta_2\) is sufficiently small, the incumbent cannot deter entry, and the outcome depends on \(c\). We call this the **defeated incumbent** subregion. In our example (i.e., \(\eta_1 = 0.5\)), when \(0.0715 \leq c < c^u(0.5) = 0.149\) the incumbent avoids capacity investment because high costs make post-entry competition unprofitable with the more capable entrant.

When \(c < 0.0715\), the incumbent and entrant are profitable in the post-entry game, corresponding to the **accommodated entry** region, which consists of the **advantage incumbent** and **advantage entrant** subregions according to which firm is more profitable.

Table 1 shows the solution’s components for the set of points identified in Figure 3. Evaluation in the larger set of points generated in the adaptive procedure described above reveals:

1. The incumbent’s capacity and nominal price are higher than the entrant’s throughout the accommodated entry region, including the advantage entrant subregion.
2. The incumbent
virtually matches the monopoly nominal price in the deterred entry subregion; we observed this for $\eta_2 = 0.5$, in Figure 3, but it was persistent for an extensive set of $\eta_2$ values. As we pointed out in §5, this implies that the threat of entry benefits customers and improves the incumbent’s revenues. (3) Near the meeting point of the deterred entry, advantage entrant, and defeated incumbent subregions (points E–H), the entrant’s profits are discontinuous while the incumbent’s are continuous and approach zero.

<p>| | | | | | | | | |</p>
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<td>0.5021</td>
<td>–</td>
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<td>–</td>
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Table 1: Capacities, prices, and profits for the points identified in Figure 3.

We have thus shown that offsetting the first mover advantage with lower variability is possible. However, perhaps with the exception of the case of a vanquished incumbent, this is unlikely, since defeating the incumbent or attaining entrant advantage requires $\eta_2 \leq 0.15$. That represents at least a 70% variability reduction relative to the incumbent, which is probably only possible via a radically different production process. Even in the more viable example of point M, an entrant with 50% of the incumbent’s variability can only expect about 25% of his opponent’s profits (Table 1). Therefore, our model suggests that in most settings, variability reduction alone is not sufficient to overcome the first mover advantage.

### 6.2 Cost Advantage

As an alternative to superior operational efficiency, an entrant might improve its competitive position via lower capacity costs. For instance, by using a proven technology, an entrant may be able to install and fine tune equipment faster and less expensively than the incumbent, who entered the market earlier with a less developed and therefore more expensive technology.
To model this situation, we analyzed equally operationally efficient firms \((\eta_1 = \eta_2 = 0.5)\) with different capacity costs. The results are analogous to §6.1, i.e., overcoming the first mover advantage with cost leadership is possible but unlikely. Using an adaptive procedure to explore the boundaries between regions, we computed the \(c_2\) boundary locations for three levels of \(c_1\), which are shown in Table 2. These clearly illustrate that while an entrant may be able to improve its competitive position moderately by enhancing its operational or cost capabilities, it can do better by improving both simultaneously.

### Table 2: Overcoming the first mover advantage with lower capacity costs \((\eta_1 = \eta_2 = 0.5)\).

<table>
<thead>
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<th>(c_1)</th>
<th>(\hat{c}_2)</th>
<th>(\hat{c}_2)</th>
<th>(\hat{c}_1)</th>
<th>(\hat{c}_1/\hat{c}_1)</th>
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6.3 Information Advantage

So far we have assumed perfect information. However, in most settings information is asymmetric: The entrant observes the incumbent and will gather at least some information on her capabilities. In contrast, the incumbent must often have to act without knowing the entrant’s identity, let alone his capabilities. To analyze the effect of the entrant’s information advantage about the opponent’s capabilities, we modify the model as follows: (1) The incumbent makes her capacity decision using forecast values \((\hat{c}_2, \hat{v}_2)\). (2) The entrant makes his capacity decision using the actual values \(c_i\) and \(v_i\) for \(i = 1, 2\). (3) The actual values are revealed to the incumbent and the firms engage in a perfect information simultaneous price subgame.

We illustrate our findings in Table 3, with the specific example of perfect cost information (i.e., \(\hat{c}_2 = c_2\), \(\hat{c}_1 = c_2 = c\), and \(av_1 = a\hat{v}_2 = 0.5\), for representative levels of \(c\). The surprise factor \(v_2/\hat{v}_2\) reflects how pleasantly (>1) or unpleasantly (<1) surprised the incumbent is about her opponent, ex post. The ratio \(\mu_1/\mu_1^*\) represents an incumbent’s capacity decision relative to her ex ante optimum. The two entries for every pair of surprise and capacity ratios are the incumbent’s and entrant’s ex post profits relative to the incumbent’s ex ante optimum, i.e., \(\Pi_i(\mu_1, v_2)/\Pi_i^*(\mu_1^*, \hat{v}_2), i = 1, 2\). Given \(c\), a column comparison for any fixed row illustrates the magnitude of the incumbent’s losses (gains) resulting from under (over) estimating her opponent’s capabilities. A row comparison for any fixed column reveals how the incumbent should modify her capacity decision to hedge against surprises.

Table 3 (a) corresponds to (ex ante) accommodated entry, and the incumbent’s deci-
Notice that in that case, the difference between the best course which shifts the system to the accommodated entry region where no amount of capacity is in profits. However, the incumbent can do nothing to protect against a 50% surprise factor, in profits, while a 25% capacity increase covers a 25% surprise factor with a 10.4% decrease in profits. However, the incumbent can do nothing to protect against a 50% surprise factor, which shifts the system to the accommodated entry region where no amount of capacity is sufficient to deter entry. Notice that in that case, the difference between the best course

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Table 3: Incumbent and entrant profits when $v_2$ is uncertain for the incumbent ($c_1 = e_2 = \hat{c}_2 = c, av_1 = a\hat{v}_2 = 0.5$). Profits relative to the incumbent’s for $v_2 = \bar{v}_2, \mu_1 = \mu_1^*$.
of action (i.e., a 5% hedge) and doing nothing is marginal. Although undesirable for the incumbent, this enhances the solution’s robustness by reducing hedging alternatives.

The above example suggests that the usefulness of information about an opponent strongly depends on market conditions; it is most valuable in environments corresponding to parameters under which the incumbent would deter entry in the perfect information case, as it can be used by the entrant to gain entry or by the incumbent to prevent it.

7 Conclusions

In this paper, we have developed an analytic model of sequential entry into a price and leadtime-sensitive market. This formulation is the first to incorporate a sequential capacity game involving firms with different operational or cost capabilities, and a subsequent price (i.e., Bertrand) subgame in which demand is determined endogenously. In contrast to previous sequential entry models, this one does not require rationing rules and its price subgame attains a pure Nash equilibrium in the region of interest. The model also contributes to the price and time competition literature by assuming firms can influence equilibrium full prices in the post-entry subgame, and by being the first to incorporate queuing and joint capacity-price decisions.

From the analysis of the model we can draw the following conclusions:

1. If the firms are equally capable, the incumbent blockades or deters entry when capacity costs are high relative to the overall demand; otherwise, she accommodates entry and her capacity, prices, and profits are always higher. Hence, the first mover advantage is either overwhelming or quite strong, depending on market conditions.

2. If the entrant has better capabilities (i.e., superior operational efficiency or capacity costs), he can defeat and even vanquish (i.e., defeat with monopoly behavior) the incumbent. When entry is accommodated, the entrant may generate higher profits than the incumbent, but his capacity and prices are always lower. Hence, it is possible to overcome the first mover advantage through better capabilities. However, as our quantitative analysis suggests, it is unlikely based on an operational efficiency or cost advantage alone, since the relative magnitudes of the necessary capabilities are high.

3. For the incumbent and entrant, gathering or concealing information on the entrant’s capabilities is most valuable when the incumbent would deter entry ex ante under conditions of perfect information, with information enabling the entrant to challenge
deterrence or the incumbent to enforce it. The incumbent, under these conditions, can use additional capacity to hedge against information uncertainty, but her hedging ability is limited when the entrant is sufficiently capable.

Although our model shows that a sufficiently capable incumbent can effectively defend a single-technology market, further research needs to focus on understanding how changing the product itself can enable entrants to break into an established market and what this implies for incumbents. Other fundamental issues remain, including: entry time decisions, dynamically evolving markets and customer loyalty effects.

Acknowledgements

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References


Appendix: Proofs

Proof of Proposition 1. Let \((m, t)\) denote money and time. There exist \(\alpha_m\) and \(\alpha_t > 0\) such that \(m = \alpha_m \tilde{m}\) and \(t = \alpha_t \tilde{t}\). Combining these identities with \(b = \Lambda - abt_0 = 1\) implies \(\alpha_t = \tilde{\Lambda} - \tilde{a} \tilde{b} \tilde{t}_0\) and \(\alpha_m = \tilde{b}/(\tilde{\Lambda} - \tilde{a} \tilde{b} \tilde{t}_0)\). The final results follow from \(a = \tilde{a} \alpha_m/\alpha_t\), \(\tilde{p} = p/\alpha_m\) and \(\tilde{\mu} = \mu \alpha_t\).

\[\text{Lemma 1.} \quad \text{If } \mu_i, \mu_j > 0 \text{ and } p_i \leq p_j \text{ then } d\lambda_i/d a < 0 \text{ for any } a > 0.\]

Proof. The market clearing conditions (5) can be recast as:

\[p_j - p_i = a (w_i - w_j), \quad (8)\]

\[1 - p_i - a w_i = \lambda_i + \lambda_j. \quad (9)\]

Let \(\lambda' \equiv \frac{d}{da}\). Differentiation of (8) and \(p_j \geq p_i\) together with (8) leads to \(w_i - w_j = a \left( \lambda'_i \partial w_j/\partial \lambda_j - \lambda'_j \partial w_i/\partial \lambda_i \right) \geq 0\) for any \(a > 0\). Combining this inequality with the derivative of (9) and the fact that \(w_i\) and \(w_j\) are strictly increasing in \(\lambda_i\) and \(\lambda_j\) yields: \(\lambda'_i \leq -w_i \partial w_j/\partial \lambda_j/ \left( \partial w_i/\partial \lambda_i + \partial w_j/\partial \lambda_j + a \partial w_i/\partial \lambda_i \partial w_j/\partial \lambda_j \right) < 0\) for any \(a > 0\). \(\square\)
Proof of Proposition 2. [1] Recall that when \( p_j \geq p^c(p_i, a v_j / \mu_j, \mu_i) \), with \( p^c(p, y, \mu) \) defined in §3.3, firm \( j \) prices herself out of the market. As \( y \to 0 \), \( p^c(p_i, y, \mu_i) \to p^c(p_i, 0, \mu_i) = [1 + p_i - \mu_i + \sqrt{(1 - p_i - \mu_i)^2}] / 2 = \max\{p_i, 1 - \mu_i\} \). Hence when the condition in [1] holds, in the limit firm \( j \) is priced out, so \( \lambda_j \to 0 \) and firm \( i \) captures the limiting monopoly demand defined in §4.1, which as \( y \to 0 \) becomes \( p^c(p_i, 0, \mu_i) \). [2] Since for every \( a > 0 \), \( \lambda_i < \mu_i \) and \( \lambda_i \) is increasing as \( a \) decreases to 0 (Lemma 1), \( \lim_{a \to 0} \lambda_i = \tilde{\lambda}_i \) exists and \( \tilde{\lambda}_i \leq \mu_i \). The left-hand-side of (8) being strictly positive and independent of \( a \), \( p^c(p_i, y, \mu_i) \to \bar{\lambda}_i \) yields: \( \lambda_i = \frac{\bar{\lambda}_i \mu_j \lambda_j}{y_j (\mu_i - \lambda_i) + y_i \bar{\lambda}_i} \), (10)

where \( y_i = v_i / \mu_i \), \( y_j = v_j / \mu_j \). Let \( \lim_{a \to 0} \lambda_i = \tilde{\lambda}_i \) and \( \lim_{a \to 0} \lambda_j = \bar{\lambda}_j \). It follows from (10) that \( \tilde{\lambda}_i = \mu_i \) iff \( \bar{\lambda}_j = \mu_j \). [a] If \( \tilde{\lambda}_i < \mu_i \) and \( \bar{\lambda}_j < \mu_j \), from (9) \( 1 - p \to \tilde{\lambda}_i + \bar{\lambda}_j < \mu_i + \mu_j \), but \( 1 - p \geq \mu_i + \mu_j \). Therefore \( \tilde{\lambda}_i = \mu_i \) and \( \bar{\lambda}_j = \mu_j \). [b] Clearly \( \tilde{\lambda}_i < \mu_i \) and \( \bar{\lambda}_j < \mu_j \); otherwise \( \tilde{\lambda}_i = \mu_i \), \( \bar{\lambda}_j = \mu_j \) and (9) imply \( 1 - p \geq \mu_i + \mu_j \). Calculating \( \tilde{\lambda}_i \) and \( \bar{\lambda}_j \) entails solving the system comprised of \( 1 - p - \lambda_i - \lambda_j = 0 \) and (10). Combining the two equations to eliminate \( \lambda_j \) yields:

\[
y_j - y_i)\lambda_i^2 - \alpha \lambda_i + (1 - p) y_j \mu_i = 0,
\]

where \( \alpha = (1 - p)(y_j - y_i) + y_j \mu_i + y_i \mu_j \). [i] If \( y_j - y_i > 0 \), \( \tilde{\lambda}_i = \frac{\alpha - \sqrt{\alpha^2 - 4 \beta}}{2(y_j - y_i)} \), where \( \beta = (y_j - y_i)(1 - p) y_j \mu_i \). Only the root with the negative radical satisfies \( \tilde{\lambda}_i < \mu_i \). Use of (10) leads to \( \tilde{\lambda}_j \). [ii] If \( y_j - y_i = 0 \) (11) becomes linear with root matching the expression in \( \gamma_i \). Use of (10) leads to the expression in \( \gamma_j \). [c] Because \( p_j = p \geq 1 - \mu_j \), it follows that \( p_i = p \geq \max\{p_j, 1 - \mu_j\} \) and firm \( i \) prices herself out of the market.

The same proof holds when \( v_i \) and \( v_j \to 0 \), provided both vanish at the same rate (e.g., \( v_i \) remaining proportional to \( v_j \) during the process).

Proof of Proposition 3. We begin with the core of the proof, which involves showing that for \( i = 1, 2 \), the revenue \( f_i(p_i) = p_i \lambda_i(p_i, p_j) \) is quasiconcave in \( p_i \). To simplify notation we define \( \eta_i = a v_i, \eta_i = \eta_i / \mu_i \) for \( i = 1, 2 \) and omit the subscript \( i \) but keep \( j \) to denote the opposite firm’s parameters ( \( j \neq i \)). Combining the market clearing equations (5) to eliminate \( \lambda_j \) yields

\[
y_j (\mu - \lambda) [(\mu - \lambda)(1 - p - \lambda) - y \lambda] + (\mu - \lambda)(p_j - p) - y \lambda [(\mu - \lambda)(-1 + p + \lambda + \mu_j) + y \lambda] = 0,
\]

on which we end the proof.
which is cubic in $\lambda$ and quadratic in $p$, so instead of working with the solution $\lambda(p)$ we use its inverse $p(\lambda)$. The roots are $p = [a(\lambda)/(\mu - \lambda) \pm \sqrt{b(\lambda)}]/2$, where $a(\lambda) = \lambda^2 - \lambda (\zeta + 2 y + \mu) + \mu \zeta$, $b(\lambda) = (\lambda - \xi)^2 + 4 \eta_j$, $\xi = 1 - \mu_j + y_j - p_j$, and $\zeta = 1 - \mu_j - y_j + p_j$. To identify the relevant root, notice from (5) that $\lambda_j = 1 - \mu - y \lambda/(\mu - \lambda)$. Using the root with the minus sign: $\lambda_j < \mu_j$ iff $(\mu - \lambda) \sqrt{b(\lambda)} < 2 (\mu_j + \lambda - 1) (\mu - \lambda) + a(\lambda) + 2 y \lambda$, but $[2 (\mu_j + \lambda - 1) (\mu - \lambda) + a(\lambda) + 2 y \lambda]^2 = (\mu - \lambda)^2 (\lambda - \xi)^2 < (\mu - \lambda)^2 b(\lambda)$, which means the relevant root is $p = a(\lambda)/2 (\mu - \lambda) + \sqrt{b(\lambda)} / 2$. The first order condition is $f'(p) = \lambda + p \lambda'(p) = 0$, and using $\lambda'(p) = 1/p'(\lambda)$ from the implicit function theorem it can be written as

$$
\psi(\lambda) := \zeta - 2 \lambda \left[ 1 + y \right. \left. \frac{2 \mu - \lambda}{(\mu - \lambda)^2} \right] = \frac{-(\lambda - \xi)^2 + 4 \eta_j + \lambda (\lambda - \xi)}{(\lambda - \xi)^2 + 4 \eta_j} := \varphi(\lambda). \quad (12)
$$

We prove $f(p)$ is pseudoconcave by showing that (12) has a unique solution. Notice that $\zeta^2 < \xi^2 + 4 \eta_j$, so $\psi(0) = \zeta > -\sqrt{\xi^2 + 4 \eta_j} = \varphi(0)$, and $\lim_{\lambda \to -\infty} \varphi(\lambda) = -\infty < \varphi(\lambda)$, so a solution exists because $\psi$ and $\varphi$ are continuous. Showing uniqueness is more involved. Notice that $\varphi'(\lambda) = -2 \left[ 1 + 2 \eta_j/ (\mu - \lambda)^3 \right] < -2$ for every $\lambda < \mu$ and $\eta > 0$. The derivatives of $\varphi$ are: $\varphi'(\lambda) = -2 (\lambda - \xi)/[(\lambda - \xi)^2 + 4 \eta_j + 4 \lambda \eta_j]/[(\lambda - \xi)^2 + 4 \eta_j]^{3/2}$, and $\varphi''(\lambda) = -12 \eta_j (\xi^2 - \lambda \xi + 4 \eta_j)/[(\lambda - \xi)^2 + 4 \eta_j]^{5/2}$. If $\lambda_0$ is a solution of (12), then

$$
\varphi(\lambda) - \psi(\lambda) = \int_{\lambda_0}^{\lambda} [\varphi'(t) - \psi'(t)] \, dt. \quad (13)
$$

We consider two cases:

(i) $\xi \leq 0$. Because for this case $\varphi''(\lambda) < 0$ everywhere, $\varphi'(\lambda) \geq \lim_{\lambda \to -\infty} \varphi'(\lambda) = -2 > \psi'(\lambda)$ for every $\lambda \geq 0$ (we compute the limit using L’Hospital’s rule). Hence, the integrand in (13) is strictly positive and $\varphi(\lambda) - \psi(\lambda) = 0$ iff $\lambda = \lambda_0$.

(ii) $\xi > 0$. We first establish several facts about $\varphi$. From an asymptotic expansion: $\lim_{\lambda \to -\infty} \varphi(\lambda)/(\xi - 2 \lambda) = 1$. In addition, $\varphi''(\lambda) = 0$ has the unique root $\lambda = \xi + 4 \eta_j/\xi$, and $\varphi''(\lambda) < 0$ for $\lambda < \xi + 4 \eta_j/\xi$ ($\lambda > \xi + 4 \eta_j/\xi$). A straightforward calculation shows that $\gamma := (\xi + 4 \eta/\xi)/2$ is the unique solution to $\varphi(\lambda) = \xi - 2 \lambda$ (i.e., the intersection of $\varphi$ with its asymptote). For $\lambda > \gamma$, $\varphi(\lambda) > \xi - 2 \lambda$, which because $\gamma$ is unique, can be shown by evaluating $\varphi(\lambda) - \xi + 2 \lambda$ elsewhere. Evaluating at the inflection point $\lambda = 2 \gamma$ yields $(8 \eta_j + \xi^2 - 4 \xi [\eta_j + 4 \eta_j/(\xi^2)]^{1/2})/\xi$, which is positive for $\xi \neq 0$. Finally, evaluating at $\gamma$, $\varphi'(\gamma) = -2 + 4 \xi^4/[\xi^2 + 4 \eta_j]^2 > -2$ for $\xi \neq 0$. We now consider two sub-cases:

(ii–a) $\zeta \leq \xi$. In this case $\psi(\lambda) < \xi - 2 \lambda$ for all $\lambda > 0$, because $\xi - 2 \lambda - \psi(\lambda) = \xi - \psi(0) + \int_0^{\lambda} \left[ (\xi - 2 t)' - \psi'(t) \right] dt > 0$, since the integrand equals $-2 - \psi'(t) > 0$ and
\[\xi - \psi(0) = \xi - \zeta \geq 0.\] Therefore, \(\psi(\lambda) < \xi - 2\lambda < \varphi(\lambda)\) for all \(\lambda > \gamma\), and any solution \(\lambda_0\) of (12) belongs to \((0, \gamma]\). But \(\gamma = (\xi + 4\eta_j)/\xi < \xi + 4\eta_j/\xi\), so \(\varphi''(\lambda) < 0\) throughout \((0, \gamma]\) which implies \(\varphi'(\lambda) \geq \varphi'(\gamma) > -2 > \psi'(\lambda)\) in the same interval. Since (13) can only have one root when \(\lambda \in (0, \gamma]\), \(\lambda_0\) is unique.

(ii–b) \(0 < \xi < \zeta\). Here a solution to the price subgame does not always exist. To establish a region where it does, given \(\eta > 0\), we find \(\overline{\mu}\) such that for every \(\eta, \eta_j \geq \eta\), and \(\mu \leq \overline{\eta}\)

\[
\psi'(2\xi/3) \leq \varphi'(\xi + 4\eta_j/\xi) \quad \text{for} \quad 0 < \xi < \zeta.
\] 

Before constructing the region, we discuss why (14) is sufficient. As can be verified by direct substitution, \(\varphi'(2\xi/3) > 0\). When \(\mu \leq 2\xi/3\), \(\varphi'(\lambda)\) is increasing in \([0, \mu)\) and therefore so is \(\varphi(\lambda) - \psi(\lambda)\), which can thus have at most one root. Now assume \(\mu > 2\xi/3\). Notice that for every \(\lambda \in [2\xi/3, \mu)\), \(\varphi'(\lambda) \leq \psi'(2\xi/3) \leq \varphi'(\xi + 4\eta_j/\xi) \leq \varphi'(\lambda)\), where the first inequality follows from \(\psi'\) being decreasing, the second is (14), and the third holds because \(\varphi'\) attains its minimum at \(\xi + 4\eta_j/\xi\). Let \(\lambda_0\) be a solution to \(\varphi(\lambda) - \psi(\lambda) = 0\). If \(\lambda_0 \in [0, 2\xi/3)\), then \(\varphi(\lambda) - \psi(\lambda) > 0\) for \(\lambda \in (\lambda_0, 2\xi/3]\) because \(\varphi - \psi\) is increasing in that region, and \(\varphi(\lambda) - \psi(\lambda) = \varphi(2\xi/3) - \psi(2\xi/3) + \int_{\xi/3}^{\lambda} [\varphi'(t) - \psi'(t)] \, dt > 0\) for \(\lambda \in (2\xi/3, \mu)\). If \(\lambda_0 \in [2\xi/3, \mu)\), then \(\varphi(\lambda) - \psi(\lambda) = 0\) iff \(\lambda = \lambda_0\) as a result of (13) with \(\lambda\) restricted to \([2\xi/3, \mu)\).

We now construct the region where (14) holds. Because \(\partial \psi'(\lambda)/\partial \eta = -4\mu/(\mu - \lambda)^3 < 0\) and \(\partial \psi'(\lambda)/\partial \mu = 4\eta(\lambda + 2\mu)/((\mu - \lambda)^4 > 0\), we have \(\psi'(2\xi/3; \eta, \mu) \leq \psi'(2\xi/3; \eta, \overline{\mu})\) for \(\eta \geq \eta_j, 2\xi/3 < \mu \leq \overline{\mu}\). In addition, \(\varphi'(\xi + 4\eta_j/\xi) \leq \varphi'(\xi + 4\eta_j/\xi)\) for \(\eta_j \geq \eta_j\), a consequence of \(\partial \varphi'(\xi + 4\eta_j/\xi)/\partial \eta_j = \xi^3/[4\eta_j(4\eta_j + \xi^2)(\eta_j + 4\eta_j^2/\xi^2)^{1/2}] > 0\) treating \(\xi\) and \(\eta_j\) as independent variables. Therefore \(\psi'(2\xi/3; \eta_j, \overline{\mu}) \leq \varphi'(\xi + 4\eta_j/\xi)\) for \(0 < \xi < \zeta\) implies (14). But \(0 < \xi < \zeta\) is equivalent to \(0 < p_j - \eta_j/\mu_j < 1 - \mu_j\), so \(\xi = 1 - \mu_j - (p_j - \eta_j/\mu_j) < 1 - \mu_j\), \(\mu_j > \eta_j/p_j > \eta_j\), and \(\xi < 1 - \mu_j < 1 - \eta_j \leq 1 - \eta_j\). Hence, it suffices to find (a large enough) \(\overline{\mu}\) such that

\[
2 + \frac{4\eta\overline{\mu}}{\overline{\mu} - 2\xi/3} - \frac{8\eta + \xi^2}{2\xi \sqrt{4\eta^2/\xi^2 + \xi^2}} \geq 0 \quad \text{for} \quad 0 < \xi < \min \left\{1 - \eta_j, \frac{3\overline{\mu}}{2} \right\},
\] 

where the left-hand-side is \(-\psi'(2\xi/3; \eta_j, \overline{\mu}) + \varphi'(\xi + 4\eta_j/\xi)\). The sought-after \(\overline{\mu}\) maximizes \(\overline{\mu}\) subject to (15). Solving numerically for different values of \(\eta\) of practical interest yields: 

\((\eta, \overline{\mu}(\eta)) = ()(0.05, 1.32), (0.075, 1.62), (0.1, 1.95), (0.2, 3.8), (0.25, 5.2), (0.3, 7.0), (0.4, 13.0), (0.5, 24.5).\)

Because these are pairs of rational numbers, we can substitute them into the left-hand-side of (15) and show, by root isolation, that it does not change sign in \(0 < \xi < 1 - \eta_j\), and by evaluation, that it is indeed positive. Notice that optimality is not necessary.

We have shown that (12) has a unique solution when \(\eta, \eta_j \geq \eta\) and \(\mu \leq \overline{\mu}\), which implies \(f(p)\) is pseudoconcave and hence quasiconcave. Existence of the pure-strategy solution fol-
Proof of Proposition 4. To show \( u_1(p_1) = p_1 \lambda_1(p_1, p_2) \) is pseudoconcave, we construct a region where the first order condition \( u_1'(p_1) = 0 \) has a unique root. Substitution of the market clearing solution and differentiation yields \( u_1'(p_1) = g(p_1)/(p_1 - p_2 + av_2/\mu_2)^2 \) with \( g(p_1) = -(2p_1 - 1)(p_1 - p_2 + av_2/\mu_2)^2 - \mu_2(p_1 - p_2)^2 - av_2(2p_1 - p_2) \), so the first order condition is equivalent to \( g(p_1) = 0 \). Notice that \( g'(p_1) = -6(p_1 - p_2 + av_2/\mu_2)(p_1 - \zeta/3) \) with \( \zeta = 1 - \mu_2 + p_2 - av_2/\mu_2 \), and \( \zeta/3 < \zeta/2 < p^c(p_2; av_2/\mu_2, \mu_2) \), so \( p_2 \geq \zeta/3 \) implies \( g(p_1) \) is decreasing in \([p_2, p^c(p_2; av_2/\mu_2, \mu_2)]\). But \( av_2 \geq 1/4 \) implies \( -\mu_2^2 + \mu_2 - av_2 \leq -\mu_2^2 + \mu_2 - 1/4 = -(\mu_2 - 1/2)^2 \leq 0 \), and hence \((-\mu_2^2 + \mu_2 - av_2)/\mu_2 \leq 0 \leq 2p_2 \), which is equivalent to \( p_2 \geq \zeta/3 \). The same conclusion follows when \( \mu_2 \geq 1 \) because that makes \(-\mu_2^2 + \mu_2 - av_2 < 0 \). For any rational \( \eta_2 = av_2 < 1/4 \) and \( \mu_2 < 1 \), the number and of roots of \( g(p_1) \) in \([p_2, 1]\) can be identified by isolation. When \( g(p_1) \) has a unique root on \([p_2, p^c(p_2; av_2/\mu_2, \mu_2)]\), \( u_1(p_1) \) is pseudoconcave and hence quasiconcave in that set. In addition, \( u_2(p_2) \) is quasiconcave on \([0, p_1]\) because it is concave on that set. Since \( u_1(p_1) \) and \( u_2(p_2) \) are well-defined and bounded on these sets, which are non-empty, closed and convex, there exists a pure-strategy Nash solution to the price subgame (Friedman 1986). Finally, from the reaction price function of firm 2, \( p^*_2(p_1) \leq p_1 \), and \( p^*_2(p_1) = p_1 \) iff \( p_1 = 0 \). Therefore, \( p^*_1 > p^*_2 \) unless \((p^*_1, p^*_2) = (0, 0)\), but when \( p_2 = 0 \), \( u_1'(0) = 1 \) which implies \( p^*_1(0) > 0 \) so \((0, 0)\) cannot be a solution. Also from firm’s 2 reaction price, \( p^*_2(p_1) = 0 \) iff \( p_1 = 0 \), so \( p^*_1 > p^*_2 = 0 \) cannot be a solution either.

Proof of Proposition 5. Let \( p^*_i = p^*_i(\mu_2) \) for \( i = 1, 2 \) be the solution to the price subgame. From Proposition 4 \( p^*_1 > p^*_2 > 0 \), but \( p^*_1 > p^*_2 \) and the market clearing conditions imply \( \lambda_2(p^*_1, p^*_2) > 0 \), so \( p^*_2 \lambda_2(p^*_1, p^*_2) > 0 \). If \( c(\mu_2) = p^*_2 \lambda_2(p^*_1, p^*_2)/\mu_2 \), then for that \( \mu_2 \), the entrant is profitable iff \( c_2 < c(\mu_2) \). Let \( \Pi(\mu_2, c_2) = p^*_2(\mu_2) \lambda_2(p^*_1(\mu_2), p^*_2(\mu_2)) - c_2 \mu_2 \), then for any \( \mu_2 > 0 \), \( \Pi(\mu^*_2, c_2(\mu^*_2)) = \max\{\Pi(\mu_2, c_2(\mu^*_2)) : \mu_2 > 0\} \geq \Pi(\mu^*_2, c_2(\mu^*_2)) > 0 \). Hence the set \( \{c > 0 : \Pi(\mu^*_2, c) > 0\} \) is non-empty and bounded from above by \( c^a(\eta_2) \) (the entrant’s critical monopoly cost). We then set \( c^\infty(\eta_2) = \sup\{c > 0 : \Pi(\mu^*_2, c) > 0\} > 0 \), which is well-defined and satisfies \( \Pi(\mu^*_2, c_2) > 0 \) iff \( c_2 < c^\infty(\eta_2) \).

Proof of Proposition 6 relies on the following sequence of lemmas about fixed-price reaction profits, defined as \( \max\{r_i(\mu_i, \mu_j) : \mu_i \geq 0\} \) for \( i = 1, 2 \).

**Lemma 2.** Firm i’s reaction profit is decreasing in \( \mu_j \), where \( j \neq i \).

**Proof.** Straightforward differentiation and inspection of the market clearing solution yields \( \partial r_i(\mu_i, \mu_j)/\partial \mu_j = \partial \lambda_i(\mu_i, \mu_j)/\partial \mu_j < 0 \). The result follows from the envelope theorem.
For every $\mu_j \geq 0$, because the supply function is convex-concave, there exists a unique inflection point for $\lambda(\cdot, \mu_j)$, denoted by $\bar{\mu}_i(\mu_j)$, and two solutions to $\lambda(\mu_i, \mu_j) = c_i$ when $c_i < \lambda(\bar{\mu}_i(\mu_j), \mu_j)$, denoted by $\mu_i^<(\mu_j)$ and $\mu_i^>(\mu_j)$ corresponding to a local minimum and maximum satisfying $\mu_i^<(\mu_j) < \bar{\mu}_i(\mu_j) < \mu_i^>(\mu_j)$. Let $c^0_i$ denote $c^0_i(v_i)$.

**Lemma 3.** When $D_0(\cdot) \equiv 0$, for every $\mu_j \geq 0$ there exists a unique critical cost $c^*_i(\mu_j)$ such that $r_i(\mu_i, \mu_j) \leq 0$ for every $\mu_i \geq 0$ iff $c_i \geq c^*_i(\mu_j)$, with (i) $0 < c^*_i(\mu_j) > c_i$ is continuous and strictly decreasing, and (ii) $c^*_i(\cdot)$ continuous and strictly decreasing, and (iii) $c^*_i(0) = c^0_i$ and $c^*_i(\mu_j) \to 0$ as $\mu_j \to \infty$. 

**Proof.** Existence and uniqueness of $c^*_i(\mu_j)$ and (i) follow from $r_i(\mu_i, \mu_j)$ being negative for $c_i = \lambda(\bar{\mu}_i(\mu_j), \mu_j)$ and becoming strictly positive as $c_i \to 0$. Notice that $r_i(\mu_i, \mu_j)$ is continuous, and from the envelope theorem, decreasing in $\mu_j$. That, and $c^*_i(\mu_j)$ being the unique solution to $r_i(\mu_i, \mu_j) = c_i$ implies (ii). The first part of (iii) follows directly from the monopoly solution. For the second part, notice that as $\mu_j \to \infty$, $\lambda_i(\mu_i, \mu_j) \to 0$ for every finite $\mu_i$, so for any $c_i > 0$, the same is true for $r_i(\mu_i, \mu_j)$. \hfill $\Box$

Lemma 3 implies: (1) when $c_i \geq c^*_i$, firm $i$ is unprofitable even as a monopolist, therefore $\mu_i^*(\mu_j) = 0$ for every $\mu_j \geq 0$; (2) for $0 < c_i < c^*_i$, $c^*_i(\mu_j) = c_i$ has a well-defined and unique solution $\mu_i^c$, representing a critical value of firm $j$’s capacity above which the profit of firm $i$ is negative or zero for every $\mu_i \geq 0$. The following lemma characterizes the structure of the entrant’s (and incumbent’s if $D_0(\cdot) \equiv 0$) reaction function.

**Lemma 4.** When $D_0(\cdot) \equiv 0$, for every $\mu_j \geq 0$ and $j \neq i$, firm $i$’s reaction is $\mu_i^*(\mu_j) = \mu_i^>(\mu_j)$ if $c_i < c^0_i$ and $\mu_j \leq \mu_i^c$, and $\mu_i^*(\mu_j) = 0$ otherwise, with $\mu_i^>(\cdot) = \max\{\mu \geq 0 : \lambda_1(\mu, \cdot) = c_i\}$ continuous on $[0, \mu_i^c]$, and $\mu_i^>(\mu_i^c) > 0$.

**Proof.** Lemma 3 addressed the case $c_i \geq c^0_i$, so let $c_i < c^0_i$. Because $\lambda_1(\cdot, \mu_j)$ is convex-concave the local maximum $\mu_i^>(\cdot)$ is continuous, and firm $i$’s optimum reaction is either $\mu_i^>(\mu_j)$ when it exists, with strictly positive profits; or the boundary $\mu_i = 0$, with zero profits. From Lemma 3, if $\mu_j < \mu_i^c$ then $c_i < c^*_i(\mu_j) < \lambda_1(\bar{\mu}_i(\mu_j), \mu_j)$, so that $\mu_i^>(\mu_j)$ is well-defined with $r_i(\mu_i^>(\mu_j), \mu_j) > 0$ and therefore $\mu_i^*(\mu_j) = \mu_i^>(\mu_j)$. For $\mu_j \geq \mu_i^c$, consider the two cases: (1) If $c_i \leq \lambda_1(\bar{\mu}_i(\mu_j), \mu_j)$, $\mu_i^>(\mu_j)$ is well-defined, $r_i(\mu_i^>(\mu_j), \mu_j) \leq 0$ so $\mu_i^*(\mu_j) = 0$; (2) if $c_i > \lambda_1(\bar{\mu}_i(\mu_j), \mu_j)$, there is no local maximum and $r_i(\cdot, \mu_j)$ is decreasing so $\mu_i^*(\mu_j) = 0$. \hfill $\Box$

Notice that in Lemmas 3 and 4, $c^*_i(\mu_1)$, $\mu_i^c$, and the entrant’s reaction function remain unchanged when $D_0(\cdot) \geq 0$, but not so for the incumbent. Also, when $\mu_1 = \mu_i^c$, the entrant is indifferent between $\mu_2 = \mu_i^c(\mu_i^c-) -$ and $\mu_2 = 0$ because its profit is zero for both choices. To avoid ambiguities we adopt the standard assumption of the entrant choosing $\mu_2 = 0$ in those cases.
Lemma 5. If \( c_1 = c_2 \), a conglomerate owning both facilities can increase its combined profits by transferring capacity from facility 2 to facility 1 iff \( \mu_1/v_1 > \mu_2/v_2 \).

Proof. The change in profitability per unit of transferred capacity is \( \partial (\lambda_1 + \lambda_2)/\partial \mu_1 - \partial (\lambda_1 + \lambda_2)/\partial \mu_2 \). For the specific demand function (4), the market clearing conditions are equivalent to \( \lambda_1 + \lambda_2 = w_i \) for \( i = 1, 2 \). Taking partial derivatives of this system with respect to \( \mu_1 \) and solving for \( \partial \lambda_1/\partial \mu_1 \) and \( \partial \lambda_2/\partial \mu_1 \) leads to \( \partial (\lambda_1 + \lambda_2)/\partial \mu_1 = - (\partial w_1/\partial \mu_1)(\partial w_2/\partial \lambda_2) \Delta^{-1} \), where \( \Delta = (1 + \partial w_1/\partial \lambda_1)(1 + \partial w_2/\partial \lambda_2) - 1 > 0 \). Similarly, \( \partial (\lambda_1 + \lambda_2)/\partial \mu_2 = - (\partial w_2/\partial \mu_2)(\partial w_1/\partial \lambda_1) \Delta^{-1} \). Therefore the above change in profitability is positive iff \( (\partial w_1/\partial \mu_1)(\partial w_2/\partial \lambda_2) < (\partial w_2/\partial \mu_2)(\partial w_1/\partial \lambda_1) \), which simplifies to \( \rho_1 > \rho_2 \) by virtue of (1). But \( w_1 = w_2 \) is equivalent to \( (\mu_1/v_1)/(\mu_2/v_2) = (\rho_1/(1 - \rho_1))/(\rho_2/(1 - \rho_2)) \), which is strictly greater than one iff \( \rho_1 > \rho_2 \).

Lemma 6. Let \( g(\cdot) \) denote the entrant’s response function. If \( D_0(\cdot) \equiv 0 \), \( v_1 \leq v_2 \), \( c_1 = c_2 \), and \( \mu_1 \) is such that \( g(\mu_1) > 0 \), there exists \( z \geq g(\mu_1) \) such that \( g(\mu_1 + z) = 0 \) and \( r_1(\mu_1 + z, 0) > r_1(\mu_1, g(\mu_1)) \).

Proof. Let \( r(\cdot, \cdot) = r_1(\cdot, \cdot) + r_2(\cdot, \cdot) \). If \( \mu_1/v_1 \geq g(\mu_1)/v_2 \), from Lemma 5 \( r_1(\mu_1 + g(\mu_1), 0) = r(\mu_1 + g(\mu_1), 0) \geq r(\mu_1, g(\mu_1)) > r_1(\mu_1, g(\mu_1)) \). Similarly, if \( \mu_1/v_1 < g(\mu_1)/v_2 \) then \( r(0, \mu_1 + g(\mu_1)) > r(0, \mu_1 + g(\mu_1)) \) therefore \( r_1(\mu_1 + g(\mu_1), 0) > r_1(\mu_1, g(\mu_1)) \). If \( g(\mu_1 + g(\mu_1)) = 0 \) then \( z = g(\mu_1) \). Otherwise, using the same procedure repeatedly yields a monotonically increasing sequence of capacities. If the sequence ever exceeds \( \mu_1^* \), \( \mu_1 + z \) is the first value exceeding \( \mu_1^* \); otherwise the sequence is bounded and thus convergent to \( \mu_1^* \) by construction, so \( \mu_1 + z = \mu_1^* \).

Proof of Proposition 6. First let \( c_1 = c_2 \). If \( \mu_1^{0*} \) denotes the incumbent’s optimal capacity for \( D_0(\cdot) \equiv 0 \), from Lemma 6 \( \mu_1^{0*} \geq \mu_1^* \). Therefore, when \( D_0(\cdot) \geq 0 \), for any \( \mu_1 < \mu_1^* \), \( \lambda(\mu_1) \tau + \lambda_1(\mu_1, g(\mu_1)) - c_1 \mu_1 \leq \lambda(\mu_1^*) \tau + \lambda_1(\mu_1, g(\mu_1^*)) - c_1 \mu_1^* \) because \( \lambda(\cdot) \) is increasing. Hence, for \( c_1 = c_2 \), \( \mu_1^* \geq \mu_1^{0*} \geq \mu_1^* \). Now let \( c_1 < c_2 \). Notice that \( \mu_1^* \) is independent of \( c_1 \), and let \( \bar{r}_1 \) denote the incumbent’s optimum revenue in the region \( \mu_1 \leq \mu_1^* \). From the envelope theorem \( d\bar{r}_1/dc_1 = \partial \bar{r}_1/\partial c_1 = \bar{\mu}_1 \). But for \( \mu_1 > \mu_1^* \), \( r_1(\mu_1, g(\mu_1)) = (\tau + 1)\lambda(\mu_1) - c_1 \mu_1 \), so \( dr_1/dc_1 = -\mu_1 \) and \( |dr_1(\mu_1, g(\mu_1))/dc_1| > |d\bar{r}_1/dc_1| \) since \( \mu_1 > \mu_1^* \geq \bar{\mu}_1 \). Therefore as \( c_1 \) decreases from \( c_2 \), the incumbent’s revenue increases more rapidly in \( \mu_1 > \mu_1^* \), so the global maximum is still attained in that region.
1 Market Clearing Conditions

We assume the expected leadtime for firm $i$ is

$$w_i = \frac{v_i}{\mu_i} \frac{\lambda_i}{\mu_i - \lambda_i} + t_0, \quad i = 1, 2.$$  \hfill (1)

The following result establishes the existence and uniqueness of a solution to the market clearing conditions

$$p_1 + a_1 w_1 = p_2 + a_2 w_2,$$ \hfill (2)

$$D(p_1 + a_1 w_1) = \lambda_1 + \lambda_2.$$ \hfill (3)

**Proposition 1.** If $p_m + a t_0 < \pi_M$ and $p_m \leq \hat{p}$, where $m = \arg\max\{p_i : i = 1, 2\}$, $\hat{p} = \arg\min\{p_i : i = 1, 2\}$, and $\gamma$ solves $D(p_m + a w_m(\gamma)) = \gamma$, there exists a unique supply vector $(\lambda_1, \lambda_2)$ that satisfies the market clearing conditions.

**Proof.** The expected leadtime $w_k$ calculated by (1) is continuous and strictly increasing in $\lambda_k$, with $w_k = t_0$ when $\lambda_k = 0$ and $w_k \to \infty$ as $\lambda_k \to \mu_k$ for $k = 1, 2$. Hence, because $p_m + a t_0 < \pi_M$, there exists $\lambda^M_m < \mu_m$ such that $p_m + a w_m(\lambda^M_m) = \pi_M$. Notice that $\gamma$ is well-defined because $D(p_m + a w_m(\lambda)) - \lambda$ is continuous and decreasing in $\lambda$, and yields $D(p_m + a t_0) > 0$ and $-\lambda^M_m < 0$ when evaluated at 0 and $\lambda^M_m$ respectively. Because $p_m + a t_0 \leq p_m + a w_m(\gamma)$, $D(p_m + a t_0) \geq D(p_m + a w_m(\gamma)) = \gamma > 0$, so $p_m + a t_0 < \pi_M$ and there exists $\lambda^M_m < \mu_m$ such that $p_m + a w_m(\lambda^M_m) = \pi_M$. The full price $\pi_k = p_k + a w_k$ is therefore continuous, strictly increasing for $\lambda_k \in [0, \lambda^M_k]$ and onto $[p_k + a t_0, \pi_M]$ for $k = 1, 2$. Since the full price $\pi_m$ for any $\lambda_m \in [0, \lambda^M_m]$ is in the range of $\pi_m$ because $[p_m + a t_0, \pi_m] \subseteq [p_m + a t_0, \pi_M]$, there exists a unique $\lambda^M_m = \lambda^M_m(\lambda_m)$ that solves (2). But as a function of $\lambda_m$, $D(p_m + a w_m) - \lambda_m - \lambda^M_m(\lambda_m)$ is continuous and strictly decreasing, with values $D(p_m + a t_0) - \lambda_m(0)$ and $-\lambda^M_m - \lambda^M_m < 0$ at 0 and $\lambda^M_m$ respectively. Therefore, it is sufficient to show $D(p_m + a t_0) \geq \lambda_m(0)$ to prove the existence and uniqueness of $\lambda_m$ solving (3). To do this, let $\lambda = \lambda_m(0)$, so $p_m + a w_m(\lambda) =
This implies $D(p_m + a w_m(\lambda)) = D(p_m + a t_0) \geq D(p_m + a w_m(\gamma))$, which in turn implies $\lambda \leq \gamma$. The desired result follows from these inequalities and $D(p_m + a w_m(\gamma)) = \gamma$. This concludes the proof.

Some further properties that are useful for characterizing the structure of the solution follow from the fact that increasing the capacity of one firm while holding everything else constant, causes the following things to happen: the equilibrium full price decreases, the total supply rate increases, the supply rate of the competitor decreases, and the supply rate of the expanding firm increases. We state these formally as:

**Proposition 2.** Let $(\lambda_1, \lambda_2)$ denote the market clearing solution and $\pi$ the equilibrium full price. Then (i) $\partial \pi / \partial \mu_1 < 0$, (ii) $\partial (\lambda_1 + \lambda_2) / \partial \mu_1 > 0$, (iii) $\partial \lambda_2 / \partial \mu_1 < 0$, and (iv) $\partial \lambda_1 / \partial \mu_1 > 0$.

**Proof.** We show (i) by contradiction. If $\partial \pi / \partial \mu_1 \geq 0$, equations (1) and (2) imply $\partial \lambda_1 / \partial \mu_1 > 0$ and $\partial \lambda_2 / \partial \mu_1 \geq 0$, but by (3) and the demand function being decreasing we obtain the desired contradiction $\partial \pi / \partial \mu_1 < 0$. Assertion (ii) follows directly from (i) and (3), while (iii) is a consequence of (i) together with (1) and (2). Finally, (ii) and (iii) imply (iv).

### 2 Simultaneous Price Subgame

In this section we assume the firms have made their capacity decisions and focus on the price subgame. Let $v_i$ and $\mu_i$ for $i = 1, 2$ be arbitrary, positive, and common knowledge. With our choice of normalized units where $b = \Lambda - a b t_0 = 1$, conditions (3) and (2) become

\[
p_i + \frac{av_i}{\mu_i} \frac{\lambda_i}{\mu_i - \lambda_i} = 1 - \lambda_1 - \lambda_2, \quad i = 1, 2. \tag{4}
\]

We define the simultaneous price subgame via the reaction price functions. Because at this stage capacity costs have been incurred, firms set prices to maximize revenues. Furthermore, since the supply rates must satisfy $0 \leq \lambda_i < \mu_i$, $i = 1, 2$ conditions (5) imply $p_i$ is limited to the interval $[0, 1]$. So for $i = 1, 2$ and $j \neq i$, the reaction function of firm $i$ given $p_j \in [0, 1]$ is defined by $\max\{p_i, \lambda_i(p_i, p_j) : 0 \leq p_i \leq 1\}$, where $\lambda_i(p_i, p_j)$ is the solution to (5) when neither firm prices itself out, the monopolist supply when only firm $j$ prices itself out, and zero otherwise.
From Proposition 1, if \( p_i \leq p^c(p_j; av_j/\mu_j, \mu_j) \), where 
\[
p^c(p; y, \mu) = \frac{1}{2} \left( 1 + p - y - \mu + \sqrt{(1 - p + y + \mu)^2 - 4(1 - p) \mu} \right),
\]
there exists a solution to (5). Since \( p_i = p^c(p_j; av_j/\mu_j, \mu_j) \) implies \( \lambda_i = 0 \) in the solution of (5), when firm \( i \) sets \( p_i > p^c(p_j; av_j/\mu_j, \mu_j) \) it prices itself out of the market and firm \( j \) (where \( j \neq i \)) becomes a monopolist. Here \( \lambda_i = 0 \) and \( \lambda_j \) is defined as the solution to 
\[
p_j + (av_j/\mu_j) \frac{\lambda_j}{(\mu_j - \lambda_j)} = 1 - \lambda_j.
\]
Clearly, any strategy in which a firm prices itself out of the market is strictly dominated by one in which it does not. Therefore, if the simultaneous game over \([0, 1] \times [0, 1]\) has a pure-strategy solution, it must lie within the region 
\[
D = \{(p_1, p_2) : 0 \leq p_1 \leq p^c(p_2; av_2/\mu_2, \mu_2), 0 \leq p_2 \leq p^c(p_1; av_1/\mu_1, \mu_1)\},
\]
and we can, without loss of generality, restrict the game to this region in which the reaction price function of firm \( i \) can be expressed as 
\[
\max \quad p_i \lambda_i(p_i, p_j) \\
\text{s.t.} \quad 0 \leq p_i \leq p^c(p_j; av_j/\mu_j, \mu_j). \tag{5}
\]

Note that the model implicitly assumes that both firms make all their installed capacity available to the market. We now show this is without loss of generality, because capacity costs are sunk and restricting the use of any portion of capacity is sub-optimal for any firm.

**Proposition 3.** It is suboptimal for the monopolist to keep any portion of her available capacity off the market.

**Proof.** The monopolist’s optimal pricing decision given available capacity \( \mu \) can be characterized by the optimization problem \( \Omega(\mu) \):
\[
\max_{p} \quad p \lambda \\
\text{s.t.} \quad p + w(\lambda|\mu) = 1 - \lambda \\
\quad \quad \quad \quad p \geq 0,
\]
where the expected delay function \( w(\lambda|\mu) \) has domain \( \lambda \in [0, \mu] \), and \( w(\lambda|\cdot) \) is decreasing for every \( \lambda > 0 \). Let \( p^* \) denote an optimal solution to \( \Omega(\mu) \), \( \lambda^* = \lambda(p^*) \), and \( r^*(\mu) = p^* \lambda^* \) the optimal revenue. Given any \( \overline{\mu} > \mu \), define \( \Delta = w(\lambda^*|\mu) - w(\lambda^*|\overline{\mu}) > 0 \). Direct substitution shows that \( (p^* + \Delta, \lambda^*) \) is feasible for \( \Omega(\overline{\mu}) \) and therefore \( r^*(\overline{\mu}) \geq (p^* + \Delta) \lambda^* > p^* \lambda^* = r^*(\mu) \). Hence, once capacity costs are sunk, constantly idling any portion of capacity (i.e., using \( \mu \) and keeping \( \overline{\mu} - \mu \) off the market) is suboptimal for the monopolist. \( \square \)
To extend the result to the duopoly, we assume firms have installed capacities, and analyze the subsequent simultaneous subgame where each firm chooses her price together with the portion of capacity she makes available to the market.

**Definition 1.** Given $\mu_1, \mu_2 > 0$, let $\Gamma(\mu_1, \mu_2)$ denote the simultaneous subgame in which for any given $\mu_j \in [0, \mu_j]$ and $p_j \geq 0$, firm $i$ ($i \neq j$) maximizes her revenues according to:

$$\max_{\mu_i, p_i} p_i \lambda_i$$

s.t. 

$$p_i + w_i(\lambda_i|\mu_i) = 1 - \lambda_i - \lambda_j$$
$$p_j + w_j(\lambda_j|\mu_j) = 1 - \lambda_i - \lambda_j$$
$$0 \leq \mu_i \leq \overline{\mu}_i$$
$$p_i \geq 0,$$

where for $i = 1, 2$, the expected delay function $w_i(\lambda|\mu)$ has domain $\lambda \in [0, \mu]$, and $w_i(\lambda|\cdot)$ is decreasing for every $\lambda > 0$. Notice that the equality constraints are the market clearing conditions.

**Proposition 4.** If $((\mu^*_1, p^*_1), (\mu^*_2, p^*_2))$ is a pure Nash equilibrium of the subgame $\Gamma(\mu_1, \mu_2)$, then $\mu^*_i = \overline{\mu}_i$ for $i = 1, 2$.

**Proof.** Let $i \neq j$. Given $(\mu_j, p_j)$, consider any $(\mu_i, p_i)$ with $\mu_i < \overline{\mu}_i$ and let $\lambda_i, \lambda_j$ be the unique solution to the market clearing equations from the subgame’s definition. The action $(\mu_i, p_i)$ is not in firm $i$’s best-response function because it is strictly dominated by $(\overline{\mu}_i, p_i + \Delta_i)$, where as in the monopoly case we define $\Delta_i = w_i(\lambda_i|\mu_i) - w_i(\lambda_i|\overline{\mu}_i) > 0$. That is the case because $(p_i + \Delta_i) + w_i(\lambda_i|\overline{\mu}_i) = p_i + w_i(\lambda_i|\mu_i) = 1 - \lambda_i - \lambda_j$, so $(\overline{\mu}_i, p_i + \Delta_i)$ is feasible (with the original $\lambda_i, \lambda_j$ solving the new market clearing equations), and $(p_i + \Delta_i) \lambda_i > p_i \lambda_i$. Therefore, since every $(\mu_i, p_i)$ action in firm $i$’s best response function satisfies $\mu_i = \overline{\mu}_i$ for $i = 1, 2$, so do $(\mu^*_1, p^*_1)$ and $(\mu^*_2, p^*_2)$.

Unfortunately, a pure-strategy solution does not always exist for subgame (6). We illustrate this with Figure 1, which depicts the reaction curves for the instance with $av_1 = 0.002$, $av_2 = 0.4$, $\mu_1 = 0.6$, and $\mu_2 = 1.8$. The dashed and solid curves represent the price reaction functions of firms 1 and 2 respectively. The discontinuity in the reaction curve of firm 2 is because of the presence of multiple local maxima in its revenue function. Failure of the reaction curves to intersect implies a pure-strategy solution does not exist for this instance of the price subgame.
3 Bounds on Competitive Behavior

3.1 Monopoly

We omit the subscript $i$ throughout this subsection. Recall that equations (5) reduce to

$$(\lambda - \mu)(\lambda + p - 1)\mu - av\lambda = 0,$$

whose only root satisfying $\lambda < \mu$ is

$$\lambda(p) := \frac{1}{2} \left[ 1 - p + \frac{av}{\mu} + \mu - \sqrt{(1 - p + \frac{av}{\mu} + \mu)^2 - 4\mu(1 - p)} \right].$$

(6)

The optimal pricing strategy $p^*(\mu)$ is chosen to maximize revenue, which is proportional to $p\lambda(p)$. Existence and uniqueness of $p^*(\mu)$ are a consequence of revenue being continuous and concave in $p$.

**Proposition 5.** The function $p\lambda(p)$ is concave in $p$.

**Proof.** The derivative of $p\lambda(p)$ can be written as $p'(\lambda)\lambda + p$ with $p(\lambda)$ the inverse of (7). Substitution of $p(\lambda) = (\lambda^2 - \lambda(\mu + av/\mu + 1) + \mu)/(\mu - \lambda)$ and differentiation with respect to $\lambda$ results in $-2[(\mu - \lambda)^3 + av\mu]/(\mu - \lambda)^3$, which is negative for $0 \leq \lambda < \mu$ because the numerator’s only real root is $\lambda = (av\mu)^{1/3} + \mu > \mu$ and the quotient is clearly negative when $\lambda = 0$. This implies the price derivative of profit is decreasing, and profit is therefore concave.

3.2 Fixed Prices

Here, we consider the special case where prices are exogenously fixed with $p_1 = p_2 = p$. Throughout this subsection, for convenience we replace the condition $\Lambda - abt_0 = 1$ with $\Lambda - abt_0 - bp = 1$ and redefine $c_i = \hat{c}_i/(p\tau_2)$ for $i = 1, 2$. 
Analysis of the fixed-price game requires a closer examination of the second mover’s reaction function $\mu_2^*(\mu_1) = \arg\max\{\lambda_2(\mu_2, \mu_1) - c_2 \mu_2 : \mu_2 \geq 0\}$, where the entrant’s supply $\lambda_2(\mu_2, \mu_1)$ corresponds to a root of the cubic equation resulting from combining equations (1)–(3) with $D(\pi) = (\Lambda - b \pi)^+$ and $p_1 = p_2$ (the only one of the three real roots such that $\lambda_2/\mu_2 < 1$ that is nonnegative). Generally, $\lambda_2(\cdot, \mu_1)$ has more than one local maximum and therefore the reaction function has multiple discontinuities. For instance, if $c_2 = 0.225$, $av_1 = 1 \times 10^{-5}$, and $v_2 = 4 v_1$, the entrant’s supply has two local maxima when $\mu_1 = 0.4$, and so the entrant’s reaction function is discontinuous for that capacity. Fortunately, as shown below, this undesirable property is restricted to extremely small variabilities, representing virtually deterministic production systems of limited practical interest.

**Proposition 6.** There exists a critical parameter $\eta^c$ such that $\lambda_i(\cdot|\mu_j)$ is convex-concave for every $\mu_j \geq 0$ when $av_j \geq \eta^c$, where $j \neq i$ and $i = 1, 2$. Furthermore, $\eta^c < 0.002$.

**Proof.** In view of Proposition 2, $\lambda_i$ is continuous and strictly increasing in $\mu_i$. It follows that the inverse $\mu_i(\lambda_i)$ is well-defined and strictly increasing. We prove the assertion by showing that $\mu_i$ is concave-convex in $\lambda_i$ when $\eta_j = av_j \geq 0.002$.

Combining equations (1) through (3) with the demand function $D(\pi) = (\Lambda - b \pi)^+$, and $p_1 = p_2$ yields the quartic $\mu_i^4 + a_3 \mu_i^3 + a_2 \mu_i^2 + a_1 \mu_i + a_0 = 0$, where $a_3 = -2 \lambda_i$, $a_2 = \lambda_i (-\lambda_i^2 + \lambda_i + \eta_i + \mu_j (1 - \lambda_i - \mu_j) \eta_i/\eta_j)/(1-\lambda_i)$, $a_1 = \lambda_i^2 (\eta_i/\eta_j) (\eta_j + \lambda_i \mu_j - \mu_j + \mu_j^2)/(1-\lambda_i)$, and $a_0 = -\lambda_i^2 \mu_j (\eta_i^2/\eta_j)/(1-\lambda_i)$. As can be verified by direct substitution, the four roots of the quartic are

$$
\mu_i = \frac{a_3}{4} \pm \frac{1}{2} \sqrt{\frac{3a_3^2}{4} - 2a_2 \pm \sqrt{z^2 - 4a_0}}
$$

where $z = \lambda_i (\eta_i/\eta_j) (-\eta_j - \lambda_i \eta_j + \mu_j - \mu_j^2)/(1-\lambda_i)$ (Abramowitz and Stegun 1972). A straightforward calculation shows that when $\eta_j > 0$, $\mu_j > 0$, and $\lambda_i < 1$, the only root satisfying $\mu_i > \lambda_i$ for $\lambda_i > 0$ corresponds to selecting both signs positive. That solution can be written as $\mu_i(\lambda_i) = \lambda_i / 2 + \sqrt{h(\lambda_i)} / 2$, with $h(\lambda) = h_1(\lambda) + h_2(\lambda) > 0$, where

$$
h_1(\lambda) = \lambda^2 - 2 \lambda \mu_j \frac{\eta_i}{\eta_j} + 2 \lambda \left( \frac{\mu_j^2}{\eta_j} \right),
$$

$$
h_2(\lambda) = 2 \frac{\lambda}{1 - \lambda} \frac{\eta_i}{\eta_j} \sqrt{4 \eta_j \mu_j^2 + (\eta_j + (1 - \lambda) \mu_j - \mu_j^2)^2}.
$$

Let $\beta(\lambda) = \eta_j + \mu_j (1 - \lambda) - \mu_j^2$, $\gamma(\lambda) = 4 \eta_j \mu_j^2 + \beta^2(\lambda)$, and $g(\lambda) = \sqrt{\gamma(\lambda)}$. Because $\mu_i''(\lambda) = h^{-3/2}(\lambda) \varphi(\lambda)/8$, with $\varphi(\lambda) = -(h')^2(\lambda) + 2 h(\lambda) h''(\lambda)$, the sign of $\mu_i''$ is completely
determined by the sign of $\varphi$. We begin by showing that $-\infty < \lim_{\lambda \to 0} \varphi(\lambda) < 0$ and $\lim_{\lambda \to -1} \varphi(\lambda) = \infty$, which implies the sign of $\mu''(\lambda)$ changes at least once as $\lambda_i$ increases. Clearly, $h(0) = 0$. Inspection of the derivatives of (8) and (9) reveals that $h''(0)$ and $h''(0)$ are finite because $g(0)$, $g'(0)$, and $g''(0)$ are because $\gamma(0) > 0$. Hence $\lim_{\lambda \to 0} \varphi(\lambda) = \lim_{\lambda \to 0} -h'(\lambda)^2$, which is finite and strictly negative because $\lim_{\lambda \to 0} h'(\lambda) = 2(\eta_i/\eta_j) (\sqrt{\gamma(0)} + \eta_j + \mu_j^2 - \mu_j)$ and $\gamma(0) = (\eta_j + \mu_j(\mu_j - 1))^2 + 4\eta_j\mu_j$. Analogously, inspecting the derivatives as $\lambda \to 1$ reveals that $\lim_{\lambda \to 1}(1 - \lambda)^4\varphi(\lambda) = 48(\eta_i/\eta_j)^2(\eta_j + \mu_j^2)^2 > 0$, so $\lim_{\lambda \to 1} \varphi(\lambda) = \infty$.

To complete the proof we show $\varphi'(\lambda) > 0$ for every $\lambda \geq 0$. It follows that $\mu''(\lambda)$ is strictly increasing with a single root and $\mu_i$ is therefore concave-convex in $\lambda_i$. Since $\varphi'(\lambda) = 2 h(\lambda) h'''(\lambda)$, it suffices to show $h'''(\lambda) > 0$.

Combining the derivatives of (8) and (9) and simplifying yields 

$$(1 - \lambda)^4 \eta_j h'''(\lambda) = 2 \eta_j f(\lambda),$$

where $f(\lambda) = 6(\eta_j + \mu_j^2 + g(\lambda)) + 6(1 - \lambda) g'(\lambda) + 3(1 - \lambda)^2 g''(\lambda) + (1 - \lambda)^3 \lambda g'''(\lambda).$ In addition, $g' = -\gamma^{-1/2} \mu_j \beta$, $g'' = 4 \gamma^{3/2} \eta_j \mu_j^4 > 0$, and $g''' = 12 \gamma^{-5/2} \eta_j \mu_j^5 \beta$. We consider three cases.

(i) $\beta \geq 0$. Since $g'' > 0$, the only negative term in $f$ is $6 (1 - q) g'$. Therefore, $f(\lambda)/6 > \mu_j^2 + g(\lambda) + (1 - \lambda) g'(\lambda) = \mu_j^2 + \gamma(\lambda)^{1/2} - (1 - \lambda) \gamma(\lambda)^{-1/2} \mu_j \beta(\lambda) \geq \mu_j^2 + \gamma(\lambda)^{1/2} - (1 - \lambda) \mu_j$ (the last inequality owed to $\gamma = 4 \eta_j \mu_j^2 + \beta^2$). When $\mu_j - (1 - \lambda) \geq 0$ the sum is clearly nonnegative. When $\mu_j - (1 - \lambda) < 0$, $(\eta_j + \mu_j(1 - \lambda) - \mu_j^2)^2 \geq (\mu_j(1 - \lambda) - \mu_j^2)^2$ and $\gamma(\lambda) \geq (\mu_j(1 - \lambda) - \mu_j^2)^2$, which implies $\gamma(\lambda)^{1/2} \geq (1 - \lambda) \mu_j - \mu_j^2$ and hence $f(\lambda) > 0$. Note that here the result holds for every $\eta_j > 0$.

(ii) $\beta < 0$ and $(\mu_j \geq 1$ or $\lambda \leq 1 - \mu_j$ or $\lambda^2 \geq 1 - \mu_j$). Here, the only negative term in $f$ is $(1 - \lambda)^3 \lambda g'''(\lambda)$, hence $f(\lambda) > 3(1 - \lambda)^2 g''(\lambda) + (1 - \lambda)^3 \lambda g'''(\lambda) = 12(1 - \lambda)^2 \eta_j \mu_j^4 \gamma^{-5/2}(\psi(\lambda),$ where $\psi(\lambda) = \gamma(\lambda) + \lambda (1 - \lambda) \mu_j \beta(\lambda)$ determines the sign of the product. For any $\lambda \geq 0$ we can write $\psi(\lambda) = \eta_j^2 + b_1 \eta_j + b_0$, where $b_1 = (1 - \lambda) (2 + \lambda) \mu_j + 2 \mu_j^2$, and $b_0 = \mu_j^4 - (1 - \lambda) (2 + \lambda) \mu_j^3 + (1 - \lambda)^2 (2 + \lambda) \mu_j^2$. As a function of $\eta_j$, $\psi(\lambda)$ is clearly convex with a minimum at $\eta_j^0 = -\mu_j (2(1 + \mu_j) - \lambda (1 + \lambda))/2$. But $\mu_j \geq 0$ and $\lambda < 1$ imply $0 < 1 + \lambda < 2$ and $\lambda < 1 + \mu_j$, therefore $\lambda (1 + \lambda) < 2 \lambda < 2 (1 + \mu_j)$, and $\eta_j^0 < 0$. It follows that $\psi(\lambda)$ is increasing in $\eta_j$ for $\eta_j \geq 0$, and at $\eta_j = 0$, $\psi(\lambda) = b_0 = \mu_j (\lambda + \mu_j - 1) (2 + \mu_j - 1)$, which is positive for $\mu_j \geq 1$ or $\lambda \leq 1 - \mu_j$ or $\lambda^2 \geq 1 - \mu_j$. Observe that as in (i), the result holds for every $\eta_j > 0$.

(iii) $\beta < 0$, $\mu_j < 1$, and $\lambda^2 < 1 - \mu_j < \lambda$. Let $S = \{(\mu_j, \lambda_i) : 0 < \mu_j < 1, \lambda_i^2 < 1 - \mu_j < \lambda_i \}$. Notice that $S \subset \{(\mu_j, \lambda_i) : 0 \leq \mu_j \leq 1, 0 \leq \lambda_i \leq 1 \}$. Since $\eta_j + \mu_j^2 \geq 0$, $f(\lambda)/3 \geq 2 g(\lambda) + 2(1 - \lambda) g'(\lambda) + (1 - \lambda)^2 g''(\lambda) + (1 - \lambda)^3 \lambda g'''(\lambda)/3 = \xi(\lambda) \gamma^{-5/2}(\lambda)$. The term $\xi(\lambda)$ can
be rearranged into $\xi(\lambda) = \sum_{n=0}^{6} b_n \eta_j^n$. We now show that with the exception of $b_1$, all other coefficients are nonnegative in $S$. Clearly $b_6 = 2$ and $b_5 = 2 \mu_j (5 (1 - \lambda_i) + 6 \mu_j)$, but $5 (1 - \lambda_i) + 6 \mu_j \geq 0$ because $1 + 6 \mu_j / 5 > \lambda_i$. The coefficient $b_4$ can be cast as $b_4 = 2 \lambda^2 - (4 + 3 \mu_j) \lambda + 2 + 3 \mu_j (1 + \mu_j)$, which has no real roots since its discriminant is $-15 \mu_j^2 < 0$. The sign of $b_4$ can thus be shown to be positive by evaluating at $\lambda = 0$. Analogously, $b_3 = 20 \mu_j^3 p(\lambda)$, where $p(\lambda) = -\lambda^3 + (3 + \mu_j) \lambda^2 - (3 + 2 \mu_j + \mu_j^2) \lambda + 1 + \mu_j + \mu_j^2 + 2 \mu_j^3$ has a single real root: $1 + (2 - 4 (2 / (47 + 3 \sqrt{249}))^{1/3} + 2^{2/3} (47 + 3 \sqrt{249})^{1/3}) \mu_j / 6 \geq 1 > \lambda$. Therefore, it does not change sign and can be shown to be positive evaluating at $\lambda = 0$. For $b_2$, $b_2 = 2 \mu_j^2 \left( [5 \mu_j^3 (\mu_j + 2 (\mu_j - 1 + \lambda))] + (1 - \lambda)^4 (5 - 2 \mu_j) \right)$. The expression in the brackets is nonnegative because of $\lambda > 1 - \mu_j$, and $5 - 2 \mu_j \geq 0$ because $\mu_j \leq 1$. Furthermore, $b_2 = 0$ iff $\mu_j = 0$. Finally, $b_0 = 2 \mu_j^3 (\mu_j - 1 + \lambda) \geq 0$ again by $\lambda > 1 - \mu_j$. It follows that for $\eta_j \geq 0$, $\xi(\lambda) \geq b_2 \eta_j^2 + b_1 \eta_j$. Notice that if $b_2 \eta^c + b_1 \geq 0$ for some $\eta^c > 0$, $b_2 \eta_j^2 + b_1 \eta_j \geq 0$ for every $\eta_j \geq \eta^c$. Since $b_2 \eta_j + b_1 \geq 0$ iff $\eta_j \geq -b_1 / b_2$, using $b_1 = 2 \mu_j^5 (\mu_j - 1 + \lambda) \zeta(\lambda)$, where $\zeta(\lambda) = (1 - \lambda)^4 (2 \mu_j - 1) + (1 - \lambda)^3 \mu_j + (1 - \lambda)^2 \mu_j^2 - 9 (1 - \lambda) \mu_j^3 + 6 \mu_j^4$, we calculate $\max \{-b_1 / b_2 : (\mu_j, \lambda_i) \in S\} \approx 0.00184$ numerically, and round up to $\eta^c = 0.002$. Notice that $\eta_j \geq \eta^c$ is only a sufficient condition for the result. Counterexamples require substantially smaller values of $\eta_j$ (e.g., 0.0001). \qed

References