Payment System Externalities and the Role of Central Bank Digital Currency *

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Abstract

We construct a model to examine how the payment processing role of banks affects their lending activity. In our model, banks operate in separate zones, and issue claims to entrepreneurs who purchase some inputs outside their own zone. Settling bank claims across zones ex post incurs a cost to a bank that is a net payer. We show that, in equilibrium, a liquidity externality arises when zones are sufficiently different in their outsourcing propensities. That is, a bank may restrict its own lending as a result of the need to hold liquidity against claims issued by another bank. The settlement cost dampens the size of this externality. We interpret wholesale Central Bank Digital Currency (CBDC) as a device that reduces the settlement cost, and show that it exacerbates lending inequalities across zones. Retail CBDC unambiguously increases lending in the model, and also increases the effectiveness of monetary policy implemented through reserves.
1 Introduction

There are large payment flows between banks. In the U.S. in 2017, the Federal Funds market had an average daily volume of U.S. $75 to $100 billion, and CHIPS, the largest private clearing house in the US, processes domestic and international transfers that amount to $1.5 trillion per day. What is the role of banks in the payment system, and how does this role affect their lending function?

To address this question, we construct a stylized model with three important elements. First, banks process payments which generate interbank flows. Second, payment flows are tied to the level of demand deposits held by banks. Some demand deposits originate from households; however, banks also create demand deposits when they issue loans. Third, banks are strategic and take these flows into account. To capture the role of the payment system in ordinary or normal times, our model has no uncertainty (and hence no bank insolvency) and identical productive opportunities across banks. As in practice, claims issued by one bank may be redeemed at another bank before being settled by the originating bank.

Our first main result is that a liquidity externality exists across banks in normal times. Because banks are connected through the payment system, claims on a bank A can be cashed in at another bank, B. Clearing and settlement are asynchronous in our model, which requires bank B to hold reserves or liquid assets against the claims of bank A in the short term. Bank B thus loses some control over its own liquidity, and, as a consequence, restricts the quantity of loans it makes. The bank thus incurs a real cost.

In our framework, banks face equally productive investment opportunities, so the first-best outcome requires equal investment across banks. However, in equilibrium, if the flow of claims is not symmetric across banks, investment is distorted away from the first-best. Suppose the claims of bank A are more likely to be redeemed at bank B than vice versa. Then, bank A lends more than the first-best level, and bank B correspondingly cuts back on loans; that is, there is cross-sectional redistribution of economic activity. Interestingly, we find that in many situations, the liquidity externality leads to bank B borrowing from bank A in the interbank market, even though, relative to the first-best level, bank B has decreased (and bank A has increased) the loans it is making to the real sector.

Both in our model and in practice, interbank settlement is costly. Many payment systems require banks to post collateral, typically proportional to payment outflows, or to prefund expected outflows. We assume in our model that net interbank claims incur a settlement cost that is borne

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1See https://apps.newyorkfed.org/markets/autorates/fed%20funds and https://www.theclearinghouse.org/payment-systems/chips

2The Bank of International Settlements Committee on Payments and Market Infrastructures provides a comprehensive guide to payment systems in different countries (see
by a bank if it is a net payer. In equilibrium, the settlement cost dampens the liquidity externality, because each bank is reluctant to be a net payer at the eventual settlement date. Of course, total welfare is reduced by the additional settlement cost.

The settlement cost is not part of the first-best calculus. In the second-best outcome, the planner too is exposed to this cost, and so has an incentive to ensure that bank payment flows are symmetric. Therefore, per dollar of investment, if bank $A$ is more likely to generate claims that are cleared by bank $B$ than vice versa, the planner reduces lending by bank $A$ and increases lending by bank $B$. Thus, the second-best planning solution yields a very different distribution of real investment than the market equilibrium.

There has been much discussion among central banks on modernizing the payment system and reducing the cost to participants. One innovation currently being considered is Central Bank Digital Currency (CBDC)\(^3\). Our model allows us to investigate the real effects of two different types of CBDC: Wholesale CBDC, which allows banks to settle claims more efficiently, and retail CBDC, which allows consumers to hold reserves with the central bank directly.

In our model, wholesale CBDC reduces the settlement cost across banks. While this increases welfare overall, it exacerbates the liquidity externality. The result is that inequality in lending across zones is increased rather than decreased. To the extent that a regulator is concerned with regional equity, this is a stark result. Retail CBDC, in contrast, has a different effect—it leads to an overall increase in lending. Interestingly, rather than replacing traditional monetary policy, we find that it in fact improves the effectiveness of monetary policy in the model.

Our model features two banks that each operate in their own local zone. Each bank makes loans to a local continuum of entrepreneurs by issuing claims, or “fountain pen money.”\(^4\) That is, the quantity of lending may exceed the amount of physical currency deposited at the bank. Entrepreneurs use these claims to purchase inputs from households for their production process. Importantly, some inputs must be outsourced; that is, purchased outside the zone in which the lending bank operates. We assume the outsourcing propensity varies across the two zones.

The need for outsourcing has two consequences. First, households holding claims on a bank in another zone may cash them in at their own local bank. This interim demand for liquidity both generates a need for interbank loans and in equilibrium constrains the amount of claims banks can issue. The interest rate in the interbank loan market adjusts to clear the market, with banks acting as price-takers in this market. Second, at the end of the game, there is a need to settle up across different zones. After netting out claims, one bank may need to transfer reserves to the other bank.

\(^3\)A brief discussion of CBDC is provided in Section 1.1.

\(^4\)This (somewhat dismissive) phrase is due to Tobin (1963).
At this point, consistent with practice, the net payer incurs a settlement cost.

There are two basic frictions in the model. First, interim liquidity needs of consumers, in conjunction with strategic behavior by banks, constrain bank lending. Second, banks face a deadweight cost to being a net payer in the payment system. This cost comprises both the collateral cost of using the current systems and the internal opportunity cost of liquidity management. The cost associated with being a net payer reduces a bank’s willingness to issue claims.

The timing of the model, with liquidity needs arising before final settlement, reflects the asynchronous nature of clearing and settlement. With many interbank transactions, settlement occurs with a lag. Sometimes, the lag is short, and may be just intra-day. In other cases, such as with banker’s acceptances or lines of credit, the lag to settlement is longer. This timing of payments matters to banks—as Afonso and Shin (2011) point out, banks generally hold low amounts of cash and reserves, and rely on the inflows from other banks to fund the the bulk of their own payment outflows.

Historically, the literature on banks has analyzed important frictions related to insolvencies and to asymmetric information. Instead of considering extreme outcomes, in this paper, we abstract away from such frictions and focus on day-to-day operations of solvent banks, and their dual role as payment processors and lenders to the real economy. In this context, we present a new liquidity externality that is a part of normal bank operations.

Our analysis differs from the seminal work of Diamond and Dybvig (1983) in two fundamental ways. First, as we are interested in frictions other than bank insolvency, our banks face no project risk, in the sense that patient consumers in our model do not have the option to withdraw their deposits at the intermediate date. Second, in the Diamond and Dybvig world, banks are simply a conduit to channel savings from households to the real economy. We allow banks to issue fountain pen money, which implies that banks must actively choose the quantity of loans they make. In the latter vein, Donaldson, Piacentino and Thakor (2018) consider the features of banks that make them the optimal entity to issue claims, and Gu, et al. (2013) show that in a model with limited commitment, banks may emerge as trustworthy agents, with claims on bank deposits being used as means of payment.

Our paper is most closely related to the literature on liquidity management in the banking system. We build on the insights provided by Freixas, Parigi, and Rochet (2000), who model an interbank payments system based on depositors having to travel to distant places to consume. Their paper focuses on the role of lines of credit in enhancing the resiliency of the banking system when a bank can be insolvent, as such contracts allow exposed banks to spread the losses to other banks. We motivate the need for interbank payments by having some producers travel to a distant location. More importantly, the main insight from our model is that the commitment inherent
in an interbank line of credit (which commit a bank to honoring claims issued by another bank) restricts lending ex ante.

Kahn and Roberds (2009) provide an introduction to the economics of payment and settlement systems in the modern economy, and highlight the role of informational frictions. Bianchi and Bigio (2018) present a calibrated general equilibrium model in which banks receive deposits, make loans and settle reserves in the interbank market. They focus on the importance of liquidity management in the presence of credit demand and other shocks. By contrast, we focus on the effect of strategic credit supply in the absence of uncertainty or informational frictions.

The costs and benefits of gross versus net settlement are examined by, among others, Kahn and Roberds (1998). Bech and Garratt (2003) consider strategic issues in intra-day liquidity management in a real-time gross settlement system. Our time horizon is somewhat longer as we consider the period over which banks issue loans to the real economy. Thus, we limit attention to net settlement in our model.

Empirically, Ashcraft, McAndrews and Skeie (2009) present a detailed analysis of the interbank market, and show that large banks typically borrow from small banks and do so as a result of liquidity shocks that arise because of large value transfers. Craig and Ma (2018) show that, in the German interbank market, some banks are persistent borrowers and others are persistent lenders, consistent with our model.

1.1 Central Bank Digital Currency

The phrase “central bank digital currency” (CBDC) is used to refer to different forms of digital fiat money. In a Bank for International Settlements (BIS) survey, Barontini and Holden (2019) mention that 70% of the world’s central banks are exploring or will soon be exploring CBDC. Bech and Garratt (2017) provide a comprehensive taxonomy that has been adapted and has come to be known as the “BIS money flower” (see Figure 1).

We use the phrase CBDC to refer to token-based digital currencies, or cryptocurrencies, issued by a central bank. As is clear from the taxonomy, there are two types of central bank cryptocurrencies: wholesale, and general purpose (or retail). Currently, CBDC projects are still at the planning stage, and large-scale adoption has not occurred in any country. Many possible variants are being discussed.\footnote{For details, see “Central Bank Digital Currencies,” Bank for International Settlements Committee on Payments and Market Infrastructures, March 2018, available at \url{https://www.bis.org/cpmi/publ/d174.pdf}.}

We therefore focus on the broad features of wholesale and retail CBDC.

Only designated financial institutions will have access to wholesale CBDC. The distinction with the current form of central bank reserves is that transactions in wholesale CBDC will be recorded on a distributed ledger. The benefit of wholesale CBDC is that interbank transactions will require
less upfront collateral, and hence may be more efficient for banks. In the context of our model, we interpret a wholesale CBDC as an innovation that reduces the settlement cost.

Most projects on wholesale CBDC have been initiated by central banks, but some are from the private sector. Among the former are Project Jasper (Bank of Canada), Project Ubin (Monetary Authority of Singapore), and the Stella Project (joint Bank of Japan and the ECB). Among the latter are “Utility Settlement Coin” by a private consortium initiated by UBS with Clearmatics.

Retail CBDC is envisioned as a new form of central bank money. Consumers will directly hold accounts at the central bank, and can transact with each other using retail CBDC. Thus, retail CBDC functions as both a store of value and a medium of exchange. Examples of retail CBDC include the e-krone in Sweden and the e-peso pilot in Uruguay. In our model, the availability of retail CBDC reduces the need for cash in the economy. Thus, households deposit less cash in banks, and also withdraw less cash to meet their interim liquidity needs.

The idea of consumers directly having accounts deposit at the central bank is mentioned by Tobin (1985). Barrdear and Kumhof (2016) find that a suitable universally accessible CBDC can raise U.S. GDP by as much as 3%, in part by reducing monetary transaction costs. Related literature on CBDC includes Rogoff (2016), who points out the usefulness of CBDC in circumventing the zero lower bound on interest rates, and Agur, Ari, and Dell' Ariccia (2019), who discuss design of an optimal CBDC in a model in which consumers can hold cash, bank deposits, and CBDC.
2 Model

Consider an economy with two suppliers of liquid claims: a non-strategic central bank and strategic
commercial banks. The central bank issues fiat money and reserves, while commercial banks issue
private money that we refer to as “bank claims.” Entrepreneurs and households constitute the real
part of the economy. Each entrepreneur and household is infinitesimal, and has to use bank claims
to secure real goods for production or consumption. In the base model, households may hold fiat
money, but do not hold reserves. In Section 4, we consider the case in which the central bank issues
a digital currency that can be directly held by households.

The main interactions in our model occur between two banks. Each bank acts as a monopolist
in its own segmented market of entrepreneurs and households. Such segmentation could arise
from either geographic differentiation or product differentiation along an unspecified dimension.
Associated with a bank is a continuum of entrepreneurs and a continuum of households, with each
having mass 1. We refer to a bank and its system of entrepreneurs and households as a zone.
To justify banks behaving as price-takers in the interbank loan market, we assume the economy
consists of a large number \( N \) of regions, or matched pairs of banks.

Each entrepreneur owns a technology which produces output \( f(k) \) from real inputs \( k \). We
assume that \( f(k) \) is strictly increasing and strictly concave, and satisfies the Inada conditions
\( \lim_{k \to 0} f'(k) = \infty \) and \( \lim_{k \to \infty} f'(k) = 0 \). The production function \( f(k) \) is the same across all
entrepreneurs and all zones. The entrepreneur is penniless, and to fund investment must borrow
from its affiliated bank. Bank claims are denoted by \( b \), and are issued to entrepreneurs as demand
deposits. They correspond to Tobin’s (1963) “fountain-pen money.” The entrepreneurs use these
claims to purchase inputs from households.

Following Freixas, Parigi and Rochet (2000), we introduce the need for an inter-zonal payment
system by assuming that entrepreneurs are assigned to buy inputs in different locations. The two
zones are differentiated by the extent of outsourcing required by their technology. The outsourcing
propensity is designated by \( \alpha \in \{ \alpha_\ell, \alpha_h \} \), where \( 0 \leq \alpha_\ell < \alpha_h \leq 1 \). We refer to a zone with
outsourcing propensity \( \alpha_i \) as zone \( i \), and to the bank in this zone as bank \( i \). The entrepreneur in
zone \( i \) needs a quantity \((1 - \alpha_i)k_i\) of inputs from their own zone and a quantity \( \alpha_i k_i \) of inputs from
zone \(-i\). For example, if \( \alpha_\ell = 0 \), entrepreneurs in zone \( \ell \) only purchase local inputs. Similarly, if
\( \alpha_h = 1 \), the entrepreneurs in zone \( h \) purchase all their inputs from zone \( \ell \).

Households in a zone initially hold fiat money, and also provide supplies to entrepreneurs. The
households deposit their fiat money in their local bank. We assume that the aggregate quantity of
fiat money in each zone is the same, \( C \). A proportion \( \lambda \) of households suffer a short-term liquidity
shock that requires a cash withdrawal before output has been produced.
Entrepreneurs purchase inputs from households in exchange for bank claims. When entrepreneurs in zone $h$ (say) purchase inputs from zone $\ell$, households in the latter zone obtain claims issued by bank $h$. These households deposit the claims into their own local bank, $\ell$. Overall, the purchasing process thus results in the claims of a bank with type $\alpha_i$ being held by a bank with type $\alpha_{-i}$ and vice versa, creating the need for an inter-zonal payment system.

In addition to issuing claims, banks maintain reserve accounts at the central bank. Following practice in the U.S., in our model cash, bank claims, and central bank reserves are all denominated in the same units (dollars) and are exchangeable at par from one form to another. We assume that a safe asset (or, equivalently, a storage technology) is available that allows a bank to transfer cash or reserves across any two points of time. There is no danger of bank insolvency, and the safe rate of return is normalized to zero, so there is no need to discount bank claims. Therefore, each unit of bank claims $b$ is priced at $1$. There is one real good that is both the input and the output, and its price is normalized to $1$. The implicit assumption here is that the input supply curve is flat at the price of $1$; assuming an upward sloping supply curve would complicate the analysis, but not qualitatively change the results.

There are three dates, $t = 0, 1, 2$. The timeline is presented in Figure 2:

- At time $t = 0$, local households deposit cash $C > 0$ in their bank, and the central bank creates reserves $R$ for each bank. These quantities, $C$ and $R$, are taken as exogenous. Each bank $i$ then makes loans in the form of take-it-or-leave-it offers to its local entrepreneurs. It does so by creating demand deposits $b_i$. Typically, the loans extended, or “inside money,” exceed the cash deposits — as discussed later, entrepreneurs’ projects can generate a surplus, creating the resources necessary to repay the higher loan amounts.

Figure 2: **Sequence of events.**

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At $t = 1$, the interbank market is active, and banks can exchange reserves with each other. This is the market for interim liquidity. We denote the interest rate (i.e., the price of borrowing and lending reserves) at which the market clears as $r$. We assume that the number of regions $N$ is large, so banks act as price-takers in this market.

Also at $t = 1$, entrepreneurs use bank claims to purchase inputs from households. Let $b_i$ denote the quantity of bank claims available to entrepreneur $i$, and $k_i$ the quantity of inputs she purchases from households. A proportion $\lambda \in [0, 1]$ of households realize a need for consumption before the output is realized. These households arrive at their bank and immediately liquidate all their bank claims (that is, claims from both bank $i$ and bank $-i$). Each bank $i$ therefore needs to either hold cash (obtained at date 0 from households) or central bank reserves to satisfy this demand. For consumption purposes, cash and central bank reserves are equivalent in this model. Further, cash and the consumption good are perfect substitutes in the model. A proportion $1 - \lambda$ of households are patient, and do not consume at this interim date.

At $t = 2$, production is realized, final interbank settlements are made, and all remaining value is remitted to households. When the output is produced, entrepreneurs deposit it back into their own bank, and the bank makes all required payments. Here, the consumption good is the same as the input good, and we assume it can be costlessly converted into reserves. If $\alpha_\ell \neq \alpha_h$, interbank settlement is required at this date. Suppose, for example, that $\alpha_h = 1$ and $\alpha_\ell = 0$. In this case, all claims issued by both banks have been turned in by households to bank $\ell$. The claims issued by bank $h$ remain liabilities of bank $h$, so bank $h$ must transfer reserves to bank $\ell$ to fulfill these claims. Further, if bank $h$ had borrowed reserves of $\$1$ from bank $-i$ at date 1, it now owes bank $\ell$ a total of $1 + r$, which also requires interbank settlement.

Observe that there are two kinds of interbank financial flows in our model. First, interbank loans are taken out at date 1 and repaid at date 2. We normalize the cost of making such transfers to zero. Second, there are flows related to the real economy, which are generated when households in zone $-i$ obtain claims issued by bank $i$. We assume that it is costly for a bank to settle such claims. Specifically, a bank that in net terms is sending funds to another bank at date 2 to settle consumer payments incurs a cost $\tau > 0$ per unit of funds transmitted.

In practice, participating in a payments system entails both explicit and implicit costs. For example, FedWire in the U.S., operated by the Federal Reserve, and CHIPS, a private clearing system, both have explicit (albeit small) fees that must be paid for using the system. More importantly, there are collateral costs as well as limits on payments. For example, FedWire requires collateral on daylight debits (i.e., net outflows), with both a fee of 50 basis points (annualized) on uncollateralized daylight overdrafts and a maximum limit on daylight overdrafts. Similarly, CHIPS has a daily pre-funding obligation based on expected net outflows.
Payment system costs are significantly higher for banks when settling consumer payments than for purely interbank transactions. As Furfine (2011) notes, banks active in the payment system send and receive payments that are 30 times larger than their reserves. Consumer payments are settled throughout the day, so that the associated collateral requirements and especially the net debit caps bite during active trading hours for banks. To facilitate this large volume of payments, the central bank provides intra-day credit to each bank. These positions are monitored per minute, and the costs are based on net payments. Thus, as in our model, the costs are incurred by net payers. The cost $\tau$ in our model represents the opportunity cost to a bank of either restricting its own activity in anticipation of future net payments or posting collateral against those payments. An indirect way to approximate the size of $\tau$ is to consider the cost of capital constraints on banks. Kisin and Manela (2016) estimate this cost to be about 30 basis points per dollar of assets.\footnote{An additional cost of being a net payer comes from the required liquidity coverage ratio—under the Basel III Accords, the liquidity coverage ratio is defined based on a bank’s future net cash outflows (see Basel Committee on Banking Supervision, 2013).}

In contrast, as mentioned by Afonso and Lagos (2014), activity in the federal funds market occurs mostly in the last two hours of the operating day (i.e., 16:30 to 18:30 eastern standard time). The loans in this market are unsecured and typically overnight, and collateral required by the payment system is returned to banks at the end of the day. Further, to the extent that some banks are persistent borrowers and others are persistent lenders in this market, the net flows are relatively small. Thus, the cost of transacting in this market is just the interest rate ($r$ in our model), and we normalize the additional cost of transferring funds to zero.

Also at date 2, final remittances are made to households. A proportion $1 - \lambda$ of households convert their bank claims into cash at this date. The final payment to households comprises the initial cash deposits as well as all profits generated by the bank.

Under autarky, a bank has total funds of $C + R$ available to pay out to impatient households at date 2. As a proportion $\lambda$ of households are impatient, it can therefore lend up to an amount $\frac{C + R}{\lambda}$ at date 1 and still meet its interim liquidity requirement. Denote $I = \frac{C + R}{\lambda}$ as the investable funds in each zone, that is, the maximal investment a bank can make under autarky. Across a pair of matched banks in a region, the total investable funds are then $2I$. Define $\beta_i = \frac{\alpha_i}{\alpha_l + \alpha_h}$ $2I$. Observe that, because $\alpha_h > \alpha_l$, we have $\beta_l < I < \beta_h$.

We impose two parameter restrictions. The first is a restriction on the marginal productivity of the technology in the economy. The second restriction ensures that the cost of interbank transfers through the payment system, $\tau$, is not prohibitively large.

**Assumption 1** The economy satisfies:

(i) Sufficiently high productivity: $f'(\beta_h) > 1$.  

\footnote{An additional cost of being a net payer comes from the required liquidity coverage ratio—under the Basel III Accords, the liquidity coverage ratio is defined based on a bank’s future net cash outflows (see Basel Committee on Banking Supervision, 2013).}
(ii) Small transaction costs: \( \tau < \min \left\{ f'(\beta_h) - 1, -f''(\beta_\ell) \alpha_\ell \right\} \).

Part (i) of the assumption ensures that in any equilibrium, all investable funds are invested in real projects rather than in the storage technology. Part (ii) is invoked in the proof of Proposition 4 to ensure that, when \( \alpha_h \) is sufficiently high compared to \( \alpha_\ell \), there exists an equilibrium in which bank \( h \) makes a settlement payment to bank \( \ell \).

2.1 Bank’s Objective Function

The bank seeks to maximize its profit. Profit is denominated in units of the consumption good at date 2, and has three components. First, the surplus from production, which is captured entirely by the bank. Each bank is a monopolist in its own zone, and so entrepreneurs are held down to their reservation utility, which is normalized to zero. Suppose bank \( i \) issues a quantity \( b_i \) of bank claims. As claims are converted at a price of 1 into real inputs, the quantity of inputs purchased, \( k_i \), equals \( b_i \). Therefore, the expected surplus from production is \( f(k_i) - k_i = f(b_i) - b_i \).

The second component of the profit function is due to interest payments in the interbank market. Let \( z_i \) denote the amount borrowed by bank \( i \) in the interbank market at date 1. Here, \( z_i < 0 \) indicates that the bank is a lender rather than a borrower. The cost of borrowing in this market is \( rz_i \), where the interest rate \( r \) is determined in equilibrium, and is taken by bank \( i \) as given.

In addition, bank \( i \) may incur a settlement cost to settle claims with bank \( -i \) at date 2. Recall that entrepreneurs in zone \( i \) have purchased a quantity \( \alpha_i b_i \) of inputs in zone \( -i \). Thus, the quantity of claims of bank \( i \) that are held by bank \( -i \) is \( \alpha_i b_i \). Similarly, \( \alpha_{-i} b_{-i} \) represents the claims of bank \( -i \) that are held by bank \( i \). Thus, the net payment between the banks as a result of such settlement is \( \alpha_i b_i - \alpha_{-i} b_{-i} \). We refer to this amount as a net transfer from bank \( i \) to bank \( -i \). If the net transfer from bank \( i \) is positive, bank \( i \) incurs an additional per unit cost of \( \tau \) to transmit it through the payment system. This is the third component of the payoff function.

Finally, note that as the return on the storage technology is zero, any amount the bank invests in this technology makes no contribution to the net profit of the bank. Putting all this together, bank \( i \)'s payoff function is

\[
\pi_i = f(b_i) - b_i - rz_i - \tau \max\{\alpha_i b_i - \alpha_{-i} b_{-i}, 0\}.
\]  

The bank faces an interim solvency constraint at date 1: It must have enough cash or reserves on hand to meet household needs at that date. Households in zone \( i \) obtain \((1 - \alpha_i)b_i\) claims from local entrepreneurs and \(\alpha_{-i}b_{-i}\) claims from distant entrepreneurs. Recall that a proportion \( \lambda \) of households cash in their claims at date 1. The demand for interim liquidity at bank \( i \) therefore amounts to \( \lambda((1 - \alpha_i)b_i + \alpha_{-i}b_{-i}) \).
Liquidity is available to bank $i$ from three sources: cash, $C$, central bank reserves, $R$, and interbank borrowing or lending, $z_i$. Recall that exogenous investable funds are represented as $I = \frac{C+R}{\lambda}$. The interim liquidity constraint for a bank at date 2 is therefore:

$$z_i + \lambda I \geq \lambda((1 - \alpha_i)b_i + \alpha_{-i}b_{-i}).$$  \hfill (2)

Any excess borrowing $z_i > \lambda((1 - \alpha_i)b_i + \alpha_{-i}b_{-i} - I)$ is invested in the safe asset from date 1 to date 2. The bank chooses $b_i$ and $z_i$ to maximize its payoff $\pi_i$ subject to the interim liquidity constraint.

The interbank rate $r$ is an important equilibrium quantity, and is determined by market-clearing in the market for reserves at date 1. Let $z_{ij}$ denote the net borrowing of bank $i$ ($i = h, \ell$) in region $j$. Summing across regions, the market-clearing constraint is:

$$\sum_{j=1}^{N} (z_{hj} + z_{\ell j}) = 0.$$  \hfill (3)

### 2.2 Planning Outcome

To establish a benchmark, we begin by considering the allocation of claims (and hence of investment and production) that would be chosen by a central planner seeking to maximize aggregate output in the economy. As each pair of matched zones is identical, we describe the planner’s problem in terms of one such pair. The outcome from a single pair is then replicated across the other $N - 1$ pairs. The planner therefore maximizes $\pi_h + \pi_\ell$, the joint profit across the two zones $h$ and $\ell$.

We consider two versions of the planner’s problem. In both versions, the planner is subject to interim liquidity requirements at each bank. In our first-best case, the planner can costlessly make transfers to settle bank-issued claims at date 2; that is, the planner is not subject to the settlement cost $\tau$. The planner’s first-best problem is:

$$\max_{\{b_h, b_\ell, z_h, z_\ell\}} f(b_h) + f(b_\ell) - b_h - b_\ell$$  \hfill (4)

subject to:

- $z_h + z_\ell = 0$
- $z_h \geq \lambda((1 - \alpha_h)b_h + \alpha_{-h}b_{-h} - I)$
- $z_\ell \geq \lambda((1 - \alpha_\ell)b_\ell + \alpha_{-\ell}b_{-\ell} - I)$

It is immediate to see that the solution to the first-best problem involves each of banks $h$ and $\ell$ issuing the same number of claims. Let the superscript $f$ denote the solution to the first-best problem.

**Lemma 1** In the planner’s solution to the first-best problem, the liquidity constraints bind, and $b_h^f = b_\ell^f = I$.  \hfill (11)
In the second-best case, in addition to the liquidity constraints, the planner is also subject to the settlement cost for bank claims, $\tau$. The objective function for the second-best problem is thus:

$$
\max_{\{b_h, b_\ell, z_h, z_\ell\}} f(b_h) + f(b_\ell) - b_h - b_\ell - \tau|\alpha_h b_h - \alpha_\ell b_\ell|.
$$

(5)

The constraints are the same as in the first-best problem.

When $\tau > 0$, in the second-best problem the planner distorts investment toward the low-outsourcing zone, to try and avoid the settlement costs at date 2. Recall that $\beta_i = \alpha_i \alpha_\ell + \alpha_\ell^2 I$, and define $\bar{\tau} = \frac{f'(\beta_\ell) - f'(\beta_h)}{\alpha_h + \alpha_\ell}$. When $\alpha_\ell > 0$, we have $\bar{\tau} \to 0$ as $\alpha_h \to \alpha_\ell$.

**Proposition 1** In the solution to the second-best planning problem, the liquidity constraints bind. Further,

(a) If $\tau < \bar{\tau}$:

(i) The solution satisfies $f'(b^*_h) - f'(b^*_\ell) = \tau(\alpha_h + \alpha_\ell)$. In particular, $\beta_h > b^*_h > b^*_\ell > \beta_\ell$, with $b^*_\ell$ increasing and $b^*_h$ decreasing in each of $\tau, \alpha_\ell, \alpha_h$.

(ii) Bank $h$ makes a net transfer to bank $\ell$ at time 2; that is, $\alpha_h b^*_h > \alpha_\ell b^*_\ell$.

(b) If $\tau \geq \bar{\tau}$, the solution has $b^*_\ell = \beta_\ell > b^*_h = \beta_h$. Here, $\alpha_h b^*_h = \alpha_\ell b^*_\ell$, so there are no net transfers at time 2.

In both cases in the proposition, the planner skews investment toward the low-outsourcing zone. If it persisted with the first-best solution of $b^*_\ell = b^*_h = I$, at date 2 it would be subject to a total settlement cost $\tau(\alpha_h - \alpha_\ell)I$. Reducing this cost requires reducing $b^*_h$ and increasing $b^*_\ell$. Relative to the first-best outcome, total output is therefore lower. The tradeoff between these two costs determines the solution to the second-best problem.

When the per unit settlement cost $\tau$ is high (part (b) of the proposition), the planner sets $\alpha_h b^*_h = \alpha_\ell b^*_\ell$ to ensure that no overall cost for payments is incurred at date 2. If the settlement cost is lower than $\bar{\tau}$, as in part (a) of the proposition, the planner adjusts investments in the two zones until the marginal benefit from the saving on the total settlement cost exactly equals the marginal cost of distorting investment away from the first-best amounts.

3 Market Equilibrium

We characterize a Nash equilibrium in the banks’ lending game, with each bank taking the interest rate in the interbank market, $r$, as given. Our implicit assumption is that $N$ is large, so that each zone is small relative to the size of the economy. We restrict attention to equilibria that are
symmetric across regions; that is, equilibria in which each high-outsourcing bank issues the same number of claims $b_h^*$ regardless of region, and similarly for each low-outsourcing bank. We refer to this as a region-symmetric equilibrium.

**Definition 1** A region-symmetric market equilibrium in the model consists of claims issued by high- and low-outsourcing banks, $b_h^*$ and $b_l^*$, net borrowing by each bank, $z_h^*$ and $z_l^*$, and an interest rate in the interbank market, $r^*$, such that:

(i) The interim liquidity constraint of each bank $i$, equation (2), is satisfied.

(ii) For each bank $i$, $b_i^*$ and $z_i^*$ maximize its payoff $\pi_i$ as shown in equation (1), given the interbank interest rate, $r^*$, and the claims issued by its matched bank, $b_{-i}^*$.

(iii) The interbank loan market clears; that is, $z_h^* + z_l^* = 0$.

### 3.1 Bank’s Best Response

In any equilibrium, the interbank interest rate $r^*$ will be strictly positive. To see this, add the net borrowing of the two banks, $z_h + z_l$, and impose the market-clearing constraint in the interbank loan market. This yields $b_h + b_l \leq 2I$. Therefore, for the loan market to clear, at least one bank must issue claims $b_i \leq I$. As $\beta_h > I$, Assumption (i) part (i) now implies that $r^* > 0$. Therefore, to determine a bank’s best response, we focus on the case that the equilibrium rate in the interbank loan market is strictly positive. Observe that at any strictly positive interest rate, the liquidity constraint for each bank $i$ will bind; that is, $z_i = \lambda((1 - \alpha_i) b_i + \alpha_{-i} b_{-i} - I)$.

At date 0, each bank $i$ chooses $b_i$, the number of claims it issues, taking as given the claims issued by bank $-i$ and the interest rate in the interbank market at date 1. As entrepreneurs are all identical, each entrepreneur in zone $i$ receives the same number of claims. Recall that there is a mass 1 of entrepreneurs; we can therefore refer to $b_i$ as the number of claims received by a single entrepreneur in zone $i$. As the input price is normalized to 1, we can equivalently think of $b_i$ as the size of the real investment made by entrepreneurs in zone $i$.

Define $g(x) = f'^{-1}(x)$. That is, $g(\cdot)$ recovers the real input level that generates a particular level of marginal product. We note that the concavity of $f(\cdot)$ implies that $g(\cdot)$ is decreasing in $x$; that is, higher marginal products are generated by lower input levels.

Two thresholds are useful in exhibiting the best response function of bank $i$. Define

$$b_i^+(r \mid \cdot) = g(1 + r\lambda(1 - \alpha_i))$$
$$b_i^-(r \mid \cdot) = g(1 + r\lambda(1 - \alpha_i) + \tau\alpha_i).$$
That is, \( b_i^+ \) denotes the investment or input level at which the marginal product in zone \( i \) is equal to \( 1 + r \lambda (1 - \alpha_i) \). Similarly, \( b_i^- \) is the investment level at which the marginal product in zone \( i \) is equal to \( 1 + r \lambda (1 - \alpha_i) + \tau \alpha_i \). As \( g(\cdot) \) is decreasing, it is immediate that when \( \tau > 0 \) we have \( b_i^+ > b_i^- \). We show that a bank which is a net receiver in the interbank payment system at date 2 will issue a quantity of claims equal to \( b_i^+ \), and correspondingly a net payer will issue claims in the amount \( b_i^- \).

The best response of bank \( i \) depends on the claims issued by bank \( -i \) in two ways. First, bank \( i \) must ensure that it has sufficient liquidity at date 1 to meet liquidity demands by local households, that hold some claims from entrepreneurs in zone \( -i \). Second, if \( \tau > 0 \) and bank \( i \) is required to make a transfer to bank \( -i \) at date 2, it incurs a transaction cost. The size of this cost depends on the relative values of \( \alpha_i b_i \) and \( \alpha_{-i} b_{-i} \). Taking both factors into account, bank \( i \)'s best response is as follows.

**Lemma 2** Suppose \( r > 0 \), and bank \( -i \) has issued claims \( b_{-i} \). The best response of bank \( i \) is:

\[
\begin{align*}
\ell^*_i &= \begin{cases} 
\ell^-_i & \text{if } \ell^-_i \geq \frac{\alpha_{-i}}{\alpha_i} b_{-i} \\
\frac{\alpha_{-i}}{\alpha_i} b_{-i} & \text{if } \frac{\alpha_{-i}}{\alpha_i} b_{-i} \in (\ell^-_i, \ell^+_i) \\
\ell^+_i & \text{if } \ell^+_i \leq \frac{\alpha_{-i}}{\alpha_i} b_{-i}
\end{cases} \\
\ell^*_i &= \lambda((1 - \alpha_i)\ell^*_i + \alpha_{-i} b_{-i} - I). \tag{9}
\end{align*}
\]

Observe that the net borrowing in the interbank market, \( \ell^*_i \), is chosen to satisfy the bank’s liquidity constraint given the claims issued by both banks, \( \ell^*_i \) and \( b_{-i} \). Thus, the bank’s optimal response when \( r > 0 \) can be reduced to the choosing the quantity of claims to issue to local entrepreneurs.

### 3.2 Liquidity Externality

As mentioned earlier, Assumption 1 ensures that the equilibrium interbank interest rate \( r^* \) is strictly positive. Therefore, the market-clearing constraint \( \ell^*_h + \ell^*_\ell = 0 \) can equivalently be written as \( \ell^*_h + \ell^*_\ell = 2I \). That is, in any market equilibrium, total investment is equal to the total investable funds across the two zones.

To highlight the liquidity externality generated by heterogeneous outsourcing propensities, we consider a special case of our model with \( \tau = 0 \); that is, one in which there are no settlement costs. In this setting, the first- and second-best planning outcomes coincide, and have \( \ell^*_\ell = \ell^*_h \). However, the market equilibrium outcome features \( \ell^*_h > \ell^*_\ell \), so that in equilibrium production is distorted toward the high-outsourcing zone. When bank \( h \) issues a greater quantity of claims, a relatively large proportion of the claims are transferred to households in zone \( \ell \). Therefore, bank \( \ell \) must hold
a greater quantity of liquid funds at date 1 to meet the short-term liquidity needs of its clients. This externality in turn inhibits the issuance of claims by bank $\ell$.

**Proposition 2** Suppose that $\tau = 0$, so that there is no settlement cost at date 2. Then, in the market equilibrium:

(i) There is a unique interest rate $r^*$.

(ii) Bank $h$ issues more claims than bank $\ell$; that is, $b^*_h > b^*_\ell$.

(iii) Bank $h$ makes a net transfer to bank $\ell$ at date 2; that is, $\alpha_h b^*_h > \alpha_\ell b^*_\ell$.

Thus, even though the technologies in the two zones have the same productivity, in a market equilibrium there can be a large dispersion in economic activity across the zones. The externality arises because aggressive lending by bank $h$ leads to higher claims deposited at bank $\ell$ at the interim date. This tightens the liquidity constraint of bank $\ell$, and inhibits its own lending. In this way, our model highlights the distinction between clearing (which occurs when bank $\ell$ pays out on behalf of bank $h$) and settlement (which occurs when bank $h$ settles up with bank $\ell$ at the end of the game). This distinction is common with many banking instruments. For example, it is a key part of intra-day liquidity management for banks. In foreign trade, banker’s acceptances are commonly cleared in a foreign country. Alternatively, one could think of bank $h$ maintaining a line of credit with bank $\ell$ and drawing it down when bank $\ell$ clears a claim it has issued.

One way to measure the extent of the liquidity externality is to consider the gap between $b^*_h$ and $b^*_\ell$. It is straightforward to show that this gap increases in the difference between $\alpha_h$ and $\alpha_\ell$. Further, note that the equilibrium entails a net transfer from bank $h$ to bank $\ell$ at date 2; that is, $\alpha_h b^*_h > \alpha_\ell b^*_\ell$. When $\tau = 0$, there is no additional cost to such a transfer.

The liquidity externality leads to bank $h$ issuing more claims than bank $\ell$; i.e., bank $h$ lends more to the real sector than bank $\ell$ does. Surprisingly, we find that in many cases, bank $h$ also lends to bank $\ell$ in the interbank market at date 1.

**Proposition 3** Suppose that $\tau = 0$ and either: (i) $f(k) = A \ln k$, where $A > 0$, or (ii) $\alpha_h \geq \frac{1}{2}$.

Then, $z_h < 0 < z_\ell$; that is, in the interbank market at date 1, bank $\ell$ borrows from bank $h$.

In the case of a general production function, if $\alpha_h < \frac{1}{2}$, bank $h$ may be either a borrower or a lender in the interbank market. For example, suppose the production function is the power function

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7 Allen and Gale (2000) and Freixas, Parigi, and Rochet (2000) highlight the importance of lines of credit in helping banks manage idiosyncratic shocks. Our model points to a cost of interbank credit lines, resulting from the liquidity externality.
\( f(k) = Ak^y \), where \( A > 0 \) and \( y \in (0, 1) \), and set \( \alpha_\ell = 0 \). We find in numerical examples that if resources in the economy are relatively abundant (that is, given other parameters, \( I \) is sufficiently high), then for all values of \( \alpha_h \), bank \( h \) is the lender in the interbank market. However, if resources are scarce (i.e., \( I \) is sufficiently low, or conversely, \( A \) is sufficiently high for a fixed value of \( I \)), for a region of \( \alpha_h \) including zero, bank \( \ell \) is the lender in the interbank market, whereas for a region of \( \alpha_h \) including \( \frac{1}{2} \), bank \( h \) is the lender.

Proposition 3 provides two circumstances in which bank \( h \) (which issues a greater quantity of claims) lends to bank \( \ell \) rather than borrows from bank \( \ell \) in the interbank market. In our model, the only purpose of interbank borrowing is to ensure that a bank has enough liquidity to pay off impatient households at date 3. Consider, for example, a situation with a high value of \( \alpha_h \), above \( \frac{1}{2} \). All else equal, if bank \( h \) expands its lending in this scenario, bank \( \ell \) needs to hold more liquidity at date 3 than bank \( h \). Thus, bank \( h \) lends cash and reserves to bank \( \ell \), rather than the other way around. That is, the bank that lends more to the real sector also lends more to the financial sector.

It is important to note that, to isolate the liquidity externality, we have assumed that the productivity of each zone is the same. If differential productivity were the only source of heterogeneity in the model, then, in equilibrium, bank \( h \) both issues more claims than bank \( \ell \) and borrows from bank \( \ell \) in the interbank market. This is the more conventional model of the interbank channel, in which it serves as a device to reallocate resources to more productive banks.

More broadly, therefore, we make the point that borrowing on the interbank market has at least two purposes: First, to fund investments, which leads more productive banks to be borrowers. Second, to fund short-term liquidity needs; these needs may not be correlated with investment productivity. In the stark case of our model, productivity is the same, but the bank that lends less to entrepreneurs has a greater need for short-term liquidity—liquidity to honor the claims of the other bank. Our model therefore suggests that to understand borrowing and lending in the interbank market, it is important to distinguish between these different motivations. In particular, bank liquidity can be induced through a liquidity externality or can be intrinsically demanded due to access to better investment opportunities.

### 3.3 Equilibrium When \( \tau > 0 \)

When the settlement cost \( \tau \) is strictly positive, the market equilibrium may be one of two kinds. In a no-payments equilibrium, neither bank makes a settlement transfer to the other bank at date 2. That is, such an equilibrium satisfies \( \alpha_\ell b_\ell^* = \alpha_h b_h^* \), or \( b_h^* = \frac{\alpha_\ell}{\alpha_h} b_\ell^* \). The second kind of equilibrium is a payments equilibrium in which \( \alpha_\ell b_\ell^* \neq \alpha_h b_h^* \). In this equilibrium, there is a net settlement transfer between banks at date 2.

Intuitively, a no-payments equilibrium exists if the outsourcing propensity \( \alpha_h \) is relatively small.
Recall from Proposition 2 that when $\tau = 0$, bank $h$ (the high-outsourcing bank) issues more claims than bank $\ell$. However, if $\tau > 0$, bank $h$ effectively suffers a penalty when it does so due to the settlement cost at date 2. If $\alpha_h$ is small enough, it prefers to issue just enough claims to ensure there are no settlement payments at date 2. As the next lemma shows, in equilibrium bank $h$ in fact issues fewer claims than bank $\ell$ in this case.

**Lemma 3** Suppose that $\tau > 0$. Then, there exists an $\bar{\alpha} \in (\alpha_\ell, 1)$ such that, if $\alpha_h < \bar{\alpha}$, then in the market equilibrium $b^*_h = \beta_\ell$ and $b^*_\ell = \beta_h$, and there are no transfers between banks at date 2. A range of interbank interest rates $r^* \in [\underline{r}, \bar{r}]$ supports this equilibrium, where $\underline{r} = \frac{f'(\beta_\ell)^{-1} - \tau \alpha_h}{\lambda (1 - \alpha_h)}$ and $\bar{r} = \frac{f'(\beta_h)^{-1}}{\lambda (1 - \alpha_\ell)}$.

Recall that $\beta_s = \frac{\alpha_s \alpha_\ell}{\alpha_h + \alpha_\ell} 2I$ for each $s = h, \ell$. Thus, $\alpha_h \beta_\ell = \alpha_\ell \beta_h$, so that if bank $h$ issues claims in the quantity $\beta_\ell$ and bank $\ell$ in the quantity $\beta_h$, there is no net settlement payment across the two banks at date 2. In Lemma 3, the lower threshold $\underline{r}$ is chosen to ensure that $b^*_h = \beta_\ell$ when $r = \underline{r}$, and the upper threshold $\bar{r}$ is chosen to ensure that $b^*_\ell = \beta_h$ when $r = \bar{r}$. As shown in the proof of the lemma, the threshold value $\bar{\alpha}$ is defined by the value of $\alpha_h$ at which $r = \bar{r}$. For small values of $\tau$, the threshold $\bar{\alpha}$ is very close to $\alpha_\ell$, as shown in Example 1 in Section 4.

In practice, payment flows among banks are large. Thus, while Lemma 3 is useful in terms of analyzing our model for all parameter values, the obverse case with $\alpha_h > \bar{\alpha}$ is of greater interest to us. When $\alpha_h > \bar{\alpha}$, the benefit to bank $h$ of issuing additional claims outweighs the settlement cost it must incur at date 2, and a payments equilibrium results.

**Proposition 4** Suppose that $\tau > 0$ and $\alpha_h > \bar{\alpha}$. Then, in the market equilibrium:

(i) The equilibrium interbank interest rate $r^*$ is unique.

(ii) $b^*_h = b^*_\ell > \beta_\ell$ and $b^*_\ell = b^*_\ell < \beta_h$.

(iii) Bank $h$ makes a net transfer to bank $\ell$ at date 2; that is, $\alpha_h b^*_h > \alpha_\ell b^*_\ell$.

Notice that, between Lemma 3 and Proposition 4, we have established that the claims issued by each bank in a market equilibrium are unique for all parameter values. In the payments equilibrium of Proposition 4, the interbank interest rate is also unique in equilibrium, whereas in the no-payments equilibrium of Lemma 3 a range of interbank interest rates supports the equilibrium. For the rest of the paper we focus on the case in which a payments equilibrium exists.

Recall from Proposition 2 that, when the settlement cost $\tau$ is zero, bank $h$ issues more claims than bank $\ell$ in a market equilibrium. In other words, the liquidity externality pushes bank $h$ toward issuing more claims than bank $\ell$, to exploit the higher outsourcing propensity of zone $h$. 

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The settlement cost, $\tau$, acts in the opposite direction, and induces bank $h$ to reduce the quantity of claims it issues. In a payments equilibrium, the overall effect from these two countervailing forces is such that, if $\alpha_h$ is close to $\bar{\alpha}$, in equilibrium bank $h$ issues fewer claims than bank $\ell$. Conversely, if $\alpha_h$ is sufficiently greater than $\bar{\alpha}$, bank $h$ issues more claims than bank $\ell$.

**Proposition 5** Suppose $\tau > 0$ and $\alpha_h > \bar{\alpha}$. Then, there exists a threshold outsourcing propensity $\hat{\alpha} \in (\bar{\alpha}, 1)$, such that if $\alpha < \hat{\alpha}$, in the market equilibrium bank $h$ issues fewer claims than bank $\ell$ (i.e., $b_h^* < b_\ell^*$), and if $\alpha > \hat{\alpha}$, bank $h$ issues more claims than bank $\ell$ (i.e., $b_h^* > b_\ell^*$).

In the proof of Proposition 5, we show that in equilibrium the quantity of claims issued by bank $h$ increases in $\alpha_h$, and conversely the quantity of claims issued by bank $\ell$ decreases in $\alpha_h$. That is, even when $\tau > 0$, the magnitude of the liquidity externality increases in the gap between the outsourcing propensities $\alpha_h$ and $\alpha_\ell$.

Finally, in parallel to Proposition 3, we show that in many cases when $\tau > 0$, bank $h$ is a lender in the interbank market at time 1. If $\alpha_h > \hat{\alpha}$, this implies that bank $h$ lends more than bank $\ell$ to the real sector, and is also a lender to the financial sector.

**Proposition 6** Suppose that $\tau > 0$ and either: (i) $\alpha_h > \bar{\alpha}$ and $f(k) = A \ln k$, where $A > 0$, or (ii) $\alpha_h \geq \max\{\frac{1}{2}, \hat{\alpha}\}$. Then, $z_h < 0 < z_\ell$; that is, in the interbank market at date 1, bank $\ell$ borrows from bank $h$.

As in the case of $\tau = 0$, with a general production function and $\alpha_h \leq \max\{\frac{1}{2}, \hat{\alpha}\}$, bank $h$ may be a lender or a borrower in the interbank market. For example, if $\alpha_\ell = 0$ and the production function is the power function $f(k) = A k^y$ with $A > 0$ and $y \in (0, 1)$, at high values of $A$ (or conversely low values of $I$), bank $h$ is the borrower in the interbank market when $\alpha_h$ is close to zero, and the lender when $\alpha_h$ is close to $\frac{1}{2}$. Once again, this highlights the differential effects of the two motives for interbank lending and borrowing: Obtaining investable funds and holding interim liquidity.

## 4 Role of Central Bank Digital Currency

### 4.1 Wholesale CBDC: Reducing Settlement Cost

In section 1.1, we introduced the notion of wholesale CBDC. Recall from Figure 1 the “money flower,” wholesale CBDC refers to token-based digital assets that are issued by the central bank. Arguably, the most successful proof of concept to date is Project Ubin, organized by the Monetary Authority of Singapore (MAS). Briefly, the MAS developed a distributed ledger system that allowed
participating banks to effect settlement using tokenized Singapore dollars. The purpose of the experiment was to build a system for interbank transfers that features improved security, reduced settlement costs, and quicker transaction-processing. These features reduce the need for upfront collateral in the payment system. Further, the use of blockchain technology allows payments to be synchronized in real time, thereby easing the restrictions imposed by net debit caps.

In our model, we interpret wholesale CBDC as a reduction in $\tau$, the settlement cost. We first consider the effect of a reduction in $\tau$ on equilibrium outcomes. We show that such a reduction exacerbates the inequality across zones. That is, the high-outsourcing zone increases its issuance of claims and the low-outsourcing zone reduces claim issuance as $\tau$ falls. Thus, the liquidity externality across zones has more bite at lower values of $\tau$.

Proposition 7 Suppose that $\alpha_h > \bar{\alpha}$. Then, in market equilibrium, a decrease in the settlement cost $\tau$ leads to:

(i) A strict decrease in the interbank interest rate, $r^*$. 

(ii) A strict increase in $b_{h^*}$, the quantity of claims issued by the high-outsourcing bank, and a strict decrease in $b_{l^*}$, the quantity of claims issued by the low-outsourcing bank.

(iii) An increase in $z_{l^*}$, the amount bank $l$ borrows from bank $h$ in the interbank market, and a corresponding decrease in $z_{h^*}$.

Proposition 7 Thus, the outsourcing propensity and the settlement cost have opposite effects in this model. All else equal, an increase in its own outsourcing propensity $\alpha_h$ leads bank $h$ to increase the quantity of its claims in equilibrium. Conversely, an increase in the settlement cost $\tau$ leads it to reduce the number of claims it issues.

To illustrate Proposition 7 and to consider the difference between equilibrium and second-best outcomes when $\tau$ is positive, we introduce a numerical example.

Example 1

Set $f(x) = 2x^{0.8}$, $\lambda = 0.2$, $C = 2$, $R = 0$, and $\alpha_x = 0.2$. We consider the bank claims issued in a payments equilibrium when $\tau = 0$ and when $\tau = 0.005$. With these parameter values, $\bar{\alpha} = 0.201$, and we let $\alpha_h$ vary from $\bar{\alpha}$ to 1. Figure 3(a) shows the claims issued by both banks in a market equilibrium for each value of $\tau$ as $\alpha_h$ varies. Figure 3(b) compares the claims issued by each bank in a market equilibrium with the claims issued in the second-best outcome when $\tau = 0.005$, again as $\alpha_h$ varies.

8Details about Project Ubin are available in the document https://www.mas.gov.sg/-/media/MAS/ProjectUbin/Project-Ubin–SGD-on-Distributed-Ledger.pdf.
In both figures, we set $f(x) = 2x^{0.8}$, $\lambda = 0.2$, $C = 2$, $R = 0$, and $\alpha_\ell = 0.2$. Figure (a) shows the claims issued by bank $h$ and bank $\ell$ in a market equilibrium with $\tau = 0$ (dashed lines) and $\tau = 50$ basis points (solid lines) as $\alpha_h$ varies. Figure (b) shows the claims issued by banks $h$ and $\ell$ in a market equilibrium (solid lines) and in the second-best outcome (dotted lines) when $\tau = 0.005$.

**Figure 3: Bank Claims Issued in Market Equilibrium and Second-best Outcome**

Two points are worth noting from Figure 3 (a). First, for both values of $\tau$, the claims issued by bank $h$ increase in $\alpha_h$. Keeping $\alpha_\ell$ fixed, an increase in $\alpha_h$ represents an increase in the liquidity externality that bank $h$ imposes on bank $\ell$. This liquidity externality has real effects, as bank $h$ invests more when $\alpha_h$ increases, so that bank $\ell$ cuts down its own investment. Second, as $\tau$ increases from 0 to 50 basis points, for each value of $\alpha_h$ bank $h$ reduces the claims it issues in equilibrium. That is, the increase in the settlement cost dampens the incentive of bank $h$ to issue claims. In the figure, when $\tau$ is set to 50 basis points, bank $h$ issues fewer claims than bank $\ell$ when $\alpha$ is less than about 0.35, and more claims than bank $\ell$ when $\alpha$ exceeds this level.

In our model, a change in $\tau$ affects the planner’s second-best solution as well as the market equilibrium. In Figure 3 (b), we plot the bank claims issued in the second-best solution and in the market equilibrium as $\alpha_h$ varies and $\tau$ is set to 50 basis points. From the figure, it is clear that in the second-best outcome, bank $\ell$ issues more claims than bank $h$ (as stated in Proposition 1). However, in the market equilibrium, we instead have bank $h$ issuing more claims than bank $\ell$ when $\alpha$ exceeds approximately 0.35.

One way to measure inter-zonal disparities in our model is to consider the difference in marginal products across the two zones. The inter-zonal productivity gap in a market equilibrium may be
defined as:

\[ \Gamma^* = |f'(b_h^*) - f'(b_\ell^*)|. \] (10)

Recall that, depending on parameter values, in a market equilibrium bank \( h \) may issue more or fewer claims than bank \( \ell \) (as shown, for example, in Figure 3). Thus, to define the equilibrium productivity gap, we consider the absolute value of the difference in the marginal products across the zones.

In the first-best planning outcome, both zones invest the same amount, so the productivity gap is zero. In the second-best outcome, zone \( h \) invests less than zone \( \ell \), and so the marginal product is higher in zone \( h \). For consistency, we define the inter-zonal productivity gap in the second-best outcome in terms of the absolute value of the difference in zone productivities. Define

\[ \Gamma^s = |f'(b_h^s) - f'(b_\ell^s)|. \] (11)

From Proposition 1, it follows that \( \Gamma^s = \tau (\alpha_h + \alpha_\ell) \), so that this gap is strictly positive when \( \tau > 0 \).

Now, suppose the introduction of wholesale CBDC leads to a reduction in the settlement \( \tau \). Recall from Proposition 4 that the second-best outcome depends on whether \( \tau \) is lower or greater than a threshold settlement cost \( \bar{\tau} \). In the second-best outcome, the inter-zonal productivity gap either decreases (when \( \tau \leq \bar{\tau} \)) or stays constant (when \( \tau > \bar{\tau} \)). In the equilibrium outcome, the inter-zonal productivity gap increases if \( b_h^* \geq b_\ell^* \) (which, from Proposition 5, occurs when \( \alpha_h > \hat{\alpha} \)), and decreases if \( b_h^* < b_\ell^* \) (which occurs when \( \alpha_h < \hat{\alpha} \)).

**Proposition 8** Suppose that \( \alpha_h > \hat{\alpha} \), and there is a small decrease in the settlement cost \( \tau \). Then:

(i) The equilibrium inter-zonal productivity gap increases if \( b_h^* \geq b_\ell^* \) and decreases if \( b_h^* < b_\ell^* \).

(ii) The inter-zonal productivity gap in the second best outcome decreases if \( \tau \leq \bar{\tau} \), and stays constant if \( \tau > \bar{\tau} \).

A reduction in the settlement cost \( \tau \) certainly mitigates the settlement friction in our model. However, in equilibrium it may exacerbates the liquidity externality. Thus, although the second-best outcome becomes closer to the first-best outcome as \( \tau \) decreases, the equilibrium outcome in fact may diverge further away from the first-best outcome.

All else equal, equilibrium welfare improves as a result of the lower settlement cost, and worsens as a result of the increase in the productivity gap. In various numerical examples, we find that overall equilibrium welfare improves when the settlement cost falls. However, the improvement in total welfare may be accompanied by increased inequality across the zones.

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4.2 Retail CBDC: Changing Banks’ Liquidity Needs

As mentioned in Section 4, retail CBDC refers to a form of digital currency that allows all consumers to have deposit accounts at the central bank. As considered by, among others, Agur, Ari, and Dell’Ariccia (2019), the ability to transact in CBDC will reduce consumers’ need for cash and bank deposits. Nevertheless, depending on the amount of CBDC issued, retail CBDC may co-exist with cash and bank deposits.

Suppose the central bank in our model issues a retail CBDC. The demand for cash for transaction purposes will decline as some consumers switch over to using the CBDC. In addition, there is a reduced need to effect transactions through the banking system, as some transactions can be carried out directly using CBDC. We assume that some consumers will continue to use cash and will continue to hold bank deposits.

Specifically, we assume that a proportion \( \gamma > 0 \) of households start using the CBDC rather than cash for transactions. As a result, at date 1, the bank receives cash deposits \((1 - \gamma)C\) from households. Further, at date 2, a fraction \((1 - \gamma)\lambda\) of households have liquidity needs that require them to cash in their bank claims. The net result of these changes is that the investable funds at a given bank are now equal to \( \hat{I}_\gamma = (1 - \gamma)C + R(1 - \gamma)\lambda \).

The introduction of a CBDC affects both the investable funds and household’s demand for liquidity from the bank. Denote \( \lambda_\gamma = (1 - \gamma)\lambda \). Then, at date 1, the interim demand for liquidity at bank \( i \) is now \( \lambda_\gamma \{(1 - \alpha_i)b_i + \alpha_{i-1}b_{i-1}\} \). Thus, the interim liquidity constraint can be written as:

\[
\hat{z}_i(\gamma) \geq \lambda_\gamma \left((1 - \alpha_i)b_i + \alpha_{i-1}b_{i-1} - \hat{I}\right). \tag{12}
\]

Define \( \beta_{\gamma h} = \frac{\alpha_h - 2I_\gamma}{\alpha_i + \alpha_h} \) and \( \beta_{\gamma \ell} = \frac{\alpha_h - 2I_\gamma}{\alpha_\ell + \alpha_h} \). As long as \( \beta_{\gamma h}, \beta_{\gamma \ell} \) and \( \tau \) satisfy Assumption [1] our previous analysis of payments equilibria goes through. Denote \( \tilde{\alpha}_\gamma \) as the threshold value of \( \alpha_h \) above which a payments equilibrium exists (as in Lemma [3] and Proposition [4]).

We show that when the central bank also has reserves, the introduction of CBDC both increases bank lending and enhances the sensitivity of investment to conventional monetary policy (i.e., increasing central bank reserves \( R \)).

**Proposition 9** Suppose that the quantity of central bank reserves is strictly positive (that is, \( C > 0 \)) and \( \beta_{\gamma h}, \beta_{\gamma \ell} \) and \( \tau \) satisfy Assumption [1]. Then, a small increase in \( \gamma \), the proportion of the households using central bank digital currency, leads to:

(i) A strict increase in the bank claims issued by both banks; that is, \( \frac{db_h^*}{d\gamma} > 0 \) and \( \frac{db_\ell^*}{d\gamma} > 0 \).

(ii) An increase in the impact of central bank reserves policy on aggregate investment; that is, \( \frac{d^2}{d\gamma d\tau}(b_h^* + b_\ell^*) > 0 \).
Thus, the co-existence of CBDC and reserves eases the liquidity constraint for a bank, and hence improves the bank’s money-creation ability. In addition, it increases the effectiveness of traditional monetary policy by making total lending more sensitive to reserves.

5 Conclusion

Ensuring a stable and efficient payment system is one of the core principles of banking supervision reforms. For example, the Basel Committee on Banking Supervision (2008) has encouraged the adoption of comprehensive rules on liquidity risk in payment systems. In this paper we highlight that a bank’s liquidity needs can depend in part on the actions of other banks in the payment system. Even though banks can freely trade reserves in an interbank market, strategic considerations affect where credit is allocated, and thus the productive efficiency of the economy. In other words, the amount a bank lends is critically affected by the fact that it functions as a part of the payment system.

In our model, the payment system creates a liquidity externality that distorts lending away from first-best levels. The settlement cost \( \tau \) dampens the extent of this externality, as the cost is incurred by the bank that is a net payer at the end of the game. Wholesale CBDC will reduce the settlement cost, but exacerbates inequalities in lending across zones. Retail CBDC, in contrast, leads to an increase in lending and also ensures that monetary policy implemented through reserves is more effective.

A standard intuition is that in the presence of an interbank market to reallocate resources, the marginal product of capital and hence investment will equalize across different production regions. As we have demonstrated, if banks are also responsible for ex post settlement in the payment system, this will not occur. Indeed, we have illustrated both cross-sectional differences in investment, and more importantly, the occurrence of aggregate differences in output from the ideal benchmark level.
Appendix: Proofs

Proof of Lemma 1

The liquidity constraints on the planner’s problem are: \( z_h \geq \lambda \left( (1 - \alpha_h) b_h + \alpha_r b_r - I \right) \) and \( z_\ell \geq \lambda \left( (1 - \alpha_\ell) b_\ell + \alpha_h b_h - I \right) \). Summing the two, we obtain \( z_h + z_\ell \geq \lambda (b_h + b_\ell - 2I) \). In conjunction with the market-clearing condition \( z_h + z_\ell = 0 \), this implies that \( b_h + b_\ell \leq 2I \).

Consider the following relaxed problem for the planner:

\[
\max_{b_h, b_\ell} f(b_h) + f(b_\ell) - b_h - b_\ell \\
\text{subject to: } b_h + b_\ell \leq 2I.
\]

At the optimum, the constraint must bind. Suppose not; then, at least one of \( b_h \) or \( b_\ell \) must be less than \( \beta_h = \frac{\alpha_\ell}{\alpha_\ell + \alpha_h}2I < 2I \). Suppose \( b_\ell < \beta_h \). From Assumption 1(i), it follows that a small increase in \( b_\ell \) strictly increases the objective function. A similar argument holds if \( b_h < \beta_h \). Thus, the constraint must bind.

As \( f(\cdot) \) is concave, and the first-best problem is completely symmetric in banks \( h \) and \( \ell \), it is now immediate that the solution involves \( b_\ell^f = b_h^f \). Now, the aggregate resource constraint is \( z_\ell + z_h = 0 \), from which it follows that it must be that \( b_\ell^f = b_h^f = I \).

Proof of Proposition 1

Consider two matched banks in the same region. The total surplus the planner generates from these two banks is

\[
\Pi = f(b_\ell) - b_\ell + f(b_h) - b_h - \tau |\alpha_\ell b_\ell - \alpha_h b_h|.
\]

(13)

The constraints are the market-clearing condition \( z_\ell + z_h = 0 \), and the liquidity constraints \( z_h \geq \lambda \left( (1 - \alpha_h) b_h + \alpha_r b_r - I \right) \) and \( z_\ell \geq \lambda \left( (1 - \alpha_\ell) b_\ell + \alpha_h b_h - I \right) \).

As in the proof of Lemma 1 consider the relaxed problem in which the liquidity constraints are replaced by the constraint \( b_h + b_\ell \leq 2I \). We first show that this constraint must bind at the optimum. Suppose not, so that \( b_h + b_\ell < 2I \). There are three cases to consider:

(i) \( \alpha_\ell b_\ell > \alpha_h b_h \). In this case, as \( \alpha_\ell < \alpha_h \), it must be that \( b_\ell < \beta_h \). Now, by Assumption 1(i), a small increase in \( b_h \) must increase the value of the objective function, as \( f(b_h) - b_h \) increases and the transaction cost term decreases.

(ii) \( \alpha_\ell b_\ell < \alpha_h b_h \). Observe that \( \beta_h + \beta_\ell = 2I \) and that \( \alpha_\ell \beta_h = \alpha_h b_h \). Thus, \( \alpha_\ell b_\ell < \alpha_h b_h \) implies that \( b_\ell < \beta_h \). Now, by Assumption 1(i), a small increase in \( b_\ell \) must increase the value of the objective function, as \( f(b_\ell) - b_\ell \) increases and the transaction cost term decreases.

(iii) \( \alpha_\ell b_\ell = \alpha_h b_h \). In this case, it must be that \( b_\ell \) and \( b_h \) are each strictly less than \( \beta_h \). Thus,
increasing \( b_\ell \) by a small amount \( \epsilon > 0 \) and \( b_h \) by an amount \( \frac{\alpha_\ell}{\alpha_h} \epsilon \) strictly increases output, and keeps the transaction cost at zero. Thus, the objective function increases.

Therefore, at the optimum, it must be that \( b_h + b_\ell = 2I \). Now, suppose that in the planner’s solution, at date 2 bank \( \ell \) needs to make a payment transfer to bank \( h \); that is, suppose \( \alpha_\ell b_\ell > \alpha_h b_h \). As \( \alpha_\ell < \alpha_h \), it must be that \( b_\ell > b_h \). Consider the following adjustment: Reduce \( b_\ell \) by a small amount \( \epsilon > 0 \) and increase \( b_h \) by \( \epsilon \). Then, total output increases at the rate \( f'(b_h) - f'(b_\ell) > 0 \). Further, the aggregate payment cost falls by \( \tau(\alpha_\ell + \alpha_h) \). Therefore, a strict improvement in the objective function is obtained, contradicting the assumption that we were at an optimum.

Hence, in the planner’s solution, it must be that \( \alpha_h b_h \geq \alpha_\ell b_\ell \). Consider some \( b_\ell \) and \( b_h \) that satisfy both the last inequality and the aggregate resource constraint \( b_\ell + b_h = 2I \). Reduce \( b_\ell \) by some small amount \( \epsilon > 0 \) and increase \( b_h \) by \( \epsilon \). The rate of change of the objective function is \( f'(b_h) - f'(b_\ell) - \tau(\alpha_h + \alpha_\ell) \). There are two possibilities:

(a) The rate of change in the objective function is strictly positive. In this case, it must be that in equilibrium \( \alpha_h b_h > \alpha_\ell b_\ell \), and the optimal solution is obtained when the rate of change of the objective function exactly hits zero; that is, at the point at which \( f'(b_h^*) - f'(b_\ell^*) = \tau(\alpha_h + \alpha_\ell) \).

(b) The rate of change in the objective function is weakly negative. In this case, the optimal solution entails \( \alpha_h b_h^* = \alpha_\ell b_\ell^* \). Using the market-clearing constraint, we obtain that \( b_\ell^* = \beta_h \) and \( b_h^* = \beta_\ell \). Hence, for this case to occur, it must be that \( f'(\beta_h) - f'(\beta_\ell) \leq \tau(\alpha_h + \alpha_\ell) \), or \( \tau \geq \frac{f'(\beta_\ell) - f'(\beta_h)}{\alpha_h + \alpha_\ell} = \bar{\tau} \).

Therefore, if \( \tau < \bar{\tau} \), we are in case (a). In this case, we have \( f'(b_h^*) - f'(b_\ell^*) = \tau(\alpha_h + \alpha_\ell) \) and \( b_\ell^* + b_h^* = 2I \). The comparative statics in part (a) (i) of the proposition now follow. Further, as \( \alpha_h b_h^* > \alpha_\ell b_\ell^* \), bank \( h \) makes a net transfer to bank \( \ell \) at time 2.

Conversely, if \( \tau \geq \bar{\tau} \), we re in case (b), so that \( b_\ell^* = \beta_h \) and \( b_h^* = \beta_\ell \). Observe that \( \alpha_h \beta_\ell = \alpha_\ell \beta_h \). Therefore, in this case, there are no net transfers between banks at time 2.

**Proof of Lemma 2**

Suppose \( r > 0 \). Then, it follows that the interim liquidity constraint in equation (2) must bind. That is, it must be the case that \( z_i = \lambda((1 - \alpha_i)b_i + b_{-i} - I) \). If not, the profit of the bank can be trivially increased by reducing \( z_i \) by a small amount.

Now, consider two matched banks \( i \) and \( -i \). Suppose first that \( \alpha_i b_i > \alpha_{-i} b_{-i} \); that is, \( b_i > \frac{\alpha_{-i}}{\alpha_i} b_{-i} \). Then, bank \( i \) is a net payer in the interbank payment system at date 2. Hence, its profit function is

\[
\pi_i = f(b_i) - b_i - r \lambda \{(1 - \alpha_i)b_i + \alpha_{-i}b_{-i} - I\} - \tau(\alpha_i b_i - \alpha_{-i} b_{-i}),
\]

(14)
where we have substituted in $z_i = \lambda((1 - \alpha)b_i + b_{-i} - I)$. The first-order condition in $b_i$ yields

$$f'(b_i) - 1 - \lambda r(1 - \alpha_i) - \tau \alpha_i = 0$$

(15)

$$b_i = g\left(1 + \lambda r(1 - \alpha_i) + \tau \alpha_i\right) \equiv b^*_i.$$  

(16)

Concavity of $f(\cdot)$ ensures that the second-order condition is satisfied. Thus, if $b^*_i > \frac{\alpha_i}{\alpha_i - 1} b_{-i}$, then $b^*_i = b_i$.

Next, suppose that $\alpha_i b_i < \alpha_i b_{-i}$; that is, $b_i > \frac{\alpha_i}{\alpha_i - 1} b_{-i}$. Then, bank $i$ is a net recipient at date 2. Hence, its profit function is

$$\pi_i = f(b_i) - b_i - r\lambda\{(1 - \alpha_i)b_i + \alpha_i b_{-i} - I\},$$

(17)

again after substituting in $z_i = \lambda((1 - \alpha)b_i + b_{-i} - I)$. The first-order condition in $b_i$ yields

$$f'(b_i) - 1 - \lambda r(1 - \alpha_i) = 0$$

(18)

$$b_i = g\left(1 + \lambda r(1 - \alpha_i)\right) \equiv b^+_i.$$  

(19)

Concavity of $f(\cdot)$ ensures that the second-order condition is satisfied. Thus, if $b^+_i < \frac{\alpha_i}{\alpha_i - 1} b_{-i}$, then $b^+_i = b_i$.

Finally, if $\frac{\alpha_i}{\alpha_i - 1} b_{-i} \in [b^-_i, b^+_i]$, then it follows that $b^*_i = \frac{\alpha_i}{\alpha_i - 1} b_{-i}$.

Given that the interim liquidity constraint binds, we immediately have $z^*_i = \lambda((1 - \alpha) b^*_i + b_{-i} - I)$.

**Proof of Proposition 2**

(i) When $\tau = 0$, we have $b^*_i = b^+_i = g(1 + r\lambda(1 - \alpha_i))$ for each bank $i = \ell, h$. That is, given $r$, the best response of bank $i$ is to set $b_i = g(1 + r\lambda(1 - \alpha_i))$. Observe that $b_i$ is strictly decreasing in $r$, and so the excess demand function $b_\ell + b_h - 2I$ is strictly decreasing in $r$.

Now, when $r = 0$ and $\tau = 0$, the profit function of bank $i$ is $\pi_i = f(b_i) - b_i$, which is independent of both $z_i$ and $b_{-i}$. Thus, with each bank optimizing, we have $b_\ell = b_h = g(1)$. As $f'(I) > 1$ by Assumption 1 part (i), at $r = 0$ there is an excess demand for funds. Observe that $b^-_i$ and $b^+_i$ are each strictly decreasing in $r$. Hence, it must be that $r > 0$. As noted in Lemma 2, this further implies that the interim liquidity constraint for each bank is binding, so that $z_i = \lambda((1 - \alpha)b_i + b_{-i} - I)$ for each $i$. The market-clearing constraint now implies that $b_h + b_\ell = 2I$.

Now, at $r = \frac{f'(I)-1}{\lambda(1-\alpha_h)}$, we have $b_h = I$. However, $b_\ell = g(1 + r\lambda(1 - \alpha_\ell)) < g(1 + r\lambda(1 - \alpha_h)) = b_h$, so at this interest rate there is excess supply of funds. Hence, there exists an interest rate $r^* \in (0, \frac{f'(I)-1}{\lambda(1-\alpha_h)})$ such that $b^*_h + b^*_\ell = 2I$. Further, as $b^*_h$ and $b^*_\ell$ are strictly decreasing in $r$, the interest rate $r^*$ is unique.
(ii) As \( r^* > 0 \), it follows that \( b_h^* = g(1 + r^* \lambda (1 - \alpha_h)) > b_\ell^* = g(1 + r^* \lambda (1 - \alpha_\ell)) \) when \( \alpha_h > \alpha_\ell \). The market-clearing constraint \( b_h^* + b_\ell^* = 2I \) now implies that \( b_h^* > I > b_\ell^* \).

(iii) As \( b_h^* > b_\ell^* \) and \( \alpha_h > \alpha_\ell \), it follows immediately that \( \alpha_h b_h^* > \alpha_\ell b_\ell^* \); that is, bank \( h \) makes a net transfer to bank \( \ell \) at time time 2.

### Proof of Proposition 3

We have \( z_h^* = \lambda \left( (1 - \alpha_h)b_h^* + \alpha_\ell b_\ell^* - I \right) \) and \( z_\ell^* = \lambda \left( (1 - \alpha_\ell)b_\ell^* + \alpha_h b_h^* - I \right) \).

Therefore, \( z_h^* > z_\ell^* \iff (1 - \alpha_\ell)b_\ell^* + \alpha_h b_h^* > (1 - \alpha_h)b_h^* + \alpha_\ell b_\ell^* \), or \( (1 - 2\alpha_\ell)b_\ell^* > (1 - 2\alpha_h)b_h^* \).

Consider the following two cases:

(i) \( \alpha_h \geq \frac{1}{2} \). If \( \alpha_h \leq \frac{1}{2} \), it is immediate that \( (1 - 2\alpha_\ell)b_\ell^* > (1 - 2\alpha_h)b_h^* \). If \( \alpha_\ell > \frac{1}{2} \), then \( \alpha_h > \alpha_\ell \implies 0 > 1 - 2\alpha_\ell > 1 - 2\alpha_h \). Now, \( b_\ell^* < b_h^* \) implies that \( (1 - 2\alpha_\ell)b_\ell^* > (1 - 2\alpha_h)b_h^* \).

Hence, for \( \alpha_h \geq \frac{1}{2} \), we have \( z_h^* > z_\ell^* \). Market-clearing now implies that \( z_h^* > 0 > z_\ell^* \).

(ii) \( \alpha_h > \frac{1}{2} \). If \( \alpha_h < \frac{1}{2} \), then \( \alpha_h > \alpha_\ell \implies 0 < 1 - 2\alpha_\ell > 1 - 2\alpha_h \). Now, \( b_\ell^* < b_h^* \) implies that \( (1 - 2\alpha_\ell)b_\ell^* > (1 - 2\alpha_h)b_h^* \).

Hence, for \( \alpha_h > \frac{1}{2} \), we have \( z_h^* > z_\ell^* \). Market-clearing now implies that \( z_h^* > 0 > z_\ell^* \).

### Proof of Lemma 3

**Step 1:** We first define \( r \) and \( \bar{r} \), and show the existence of a \( \bar{\alpha} \) at which \( r(\bar{\alpha}) = \bar{r}(\bar{\alpha}) \).

In what follows, we keep \( \alpha_\ell \) fixed and vary \( \alpha_h \). For notational convenience, in this step alone, let \( \alpha \) denote \( \alpha_h \). Then, \( \beta_\ell = \frac{2\alpha_\ell}{\alpha_\ell + \alpha}I \) and \( \beta_h = \frac{2\alpha}{\alpha + \alpha_\ell}I \).

Define

\[
r(\alpha \mid \alpha_\ell, \tau, I) = \frac{f'(\beta_\ell) - 1 - \tau \alpha}{\lambda(1 - \alpha)}
\]

(20)

\[
\bar{r}(\alpha \mid \alpha_\ell, \tau, I) = \frac{f'(\beta_h) - 1}{\lambda(1 - \alpha_\ell)}
\]

(21)

Then, \( r \) is the interest rate at which \( b_h^- = \beta_\ell \). If \( r > r \), we have \( b_h^- < \beta_\ell \) and if \( r < r \) we have \( b_h^- > \beta_\ell \). Similarly, \( \bar{r} \) is the interest rate at which \( b_h^+ = \beta_h \), with \( b_h^+ > \beta_h \) when \( r < \bar{r} \) and \( b_h^+ < \beta_h \) when \( r > \bar{r} \). Observe that \( r(\alpha) > 0 \) as \( f'(\beta_h) > 1 \) (by Assumption 1 (i)), and \( r(\alpha) > 0 \) as \( f'(\beta_\ell) > f'(\beta_h) > 1 + \tau \geq 1 + \tau \alpha \), where \( f'(\beta_h) > 1 + \tau \) follows from Assumption 1 (ii).

Suppose that \( \alpha = \alpha_\ell \); i.e., the outsourcing propensities are the same across the two banks. Observe that \( r(\alpha \mid \cdot) \leq \bar{r}(\alpha \mid \cdot) \), with strict inequality when \( \tau > 0 \) and \( \alpha_\ell > 0 \).
Now, consider $\alpha$ increasing, starting at $\alpha = \alpha_t$. As $\frac{2\alpha}{\alpha + \alpha_t}$ is increasing in $\alpha$ and $g(\cdot)$ is a decreasing function, it follows that $\bar{r}(\alpha | \cdot) \leq \bar{r}(\alpha_t | \cdot)$. The partial derivative of $\bar{r}(\alpha)$ with respect to $\alpha$ is

$$\frac{\partial \bar{r}(\alpha)}{\partial \alpha} = \frac{(1 - \alpha)( - f''(\beta_{\ell}) \frac{\beta_I}{\alpha + \alpha_t} - \tau) + \left( f'(\beta_{\ell}) - 1 - \tau \alpha \right)}{\lambda(1 - \alpha)^2} \quad (22)$$

Now, under Assumption 1 part (ii), we have $\tau < -f''(\beta_{\ell}) \frac{\beta_I}{\alpha + \alpha_t}$ and $f'(\beta_{\ell}) > 1 + \tau$. The left-hand side of the latter inequality is weakly less than $f'(\beta_{\ell})$, and the right-hand side is strictly greater than $1 + \tau \alpha$. Therefore, both terms contained in the large parentheses in the numerator of equation (22) are strictly positive, so that $\bar{r}(\alpha_h | \cdot)$ is strictly increasing in $\alpha_h$.

Finally, observe that as $\alpha_h \to 1$, $\bar{r}(\alpha | \cdot) \to \infty$, whereas $\bar{r}(\alpha | \cdot)$ stays finite.

It now follows that there exists an $\bar{\alpha} \in (\alpha_t, 1)$ such that $\bar{r}(\alpha | \cdot) \leq \bar{r}(\alpha | \cdot)$ for $\alpha \leq \bar{\alpha}$, with $\bar{r}(\alpha | \cdot) > \bar{r}(\alpha | \cdot)$ for $\alpha_h > \bar{\alpha}$.

Step 2: We now show that $\alpha_h \leq \bar{\alpha}$ implies a no-payments equilibrium.

Suppose that $\alpha_h < \bar{\alpha}$, so that $\bar{r}(\alpha_h | \cdot) \leq \bar{r}(\alpha_{\bar{\alpha}})$. Consider any $r \in [r, \bar{r}]$. We have shown that, for such an $r$, we have $b^-_{\ell} \leq \beta_{\ell} < \bar{\beta}_{\ell} \leq b^+_{\ell}$. Observe that (i) $\alpha_{\ell} \bar{\beta}_{\ell} = \alpha_{h} \beta_{\ell}$ and (ii) $\bar{\beta}_{\ell} + \beta_{\ell} = 2I$.

Set $b^+_{\ell} = \bar{\beta}_{\ell}$ and $b^-_{\ell} = \beta_{\ell}$. Because $\bar{\beta}_{\ell} + \beta_{\ell} = 2I$, the market-clearing condition is satisfied.

Now, observe that $\bar{r}(\alpha_h) > 0$, as part (ii) of Assumption 1 implies that $f'(\beta_{\ell}) > 1 + \tau \alpha$. Hence, $r > 0$, which implies that $b^-_{\ell} > b^-_{\ell}$ and $b^+_{\ell} > b^+_{\ell}$. Therefore, we have $b^+_{\ell} = \beta_{\ell} \in (b^-_{\ell}, b^+_{\ell})$ and $b^-_{\ell} = \beta_{\ell} \in [b^-_{\ell}, b^+_{\ell})$. Therefore, from Lemma 2 each bank is playing a best response, and hence we have a Nash equilibrium in the banks' game.

Finally, we show that there is no equilibrium with $r < \bar{r}(\alpha_h) \text{ or } r > \bar{r}(\alpha_h)$. Suppose there is an equilibrium with $r < \bar{r}(\alpha_h)$. As mentioned earlier, $r < \bar{r}(\alpha_h) \implies b^+_{\ell} > \beta_{\ell}$. From Lemma 2 we have $b^+_{\ell} \in [b^-_{\ell}, b^+_{\ell}]$, so in this case it must be that $b^+_{\ell} > \beta_{\ell}$. Now, the market-clearing constraint implies that $b^+_{\ell} < \beta_{\ell}$. Hence, as $\alpha_{\ell} \beta_{\ell} = \alpha_{h} \beta_{\ell}$, it must be that $\alpha_{\ell} b^+_{\ell} < \alpha_{h} b^-_{\ell}$. But then Lemma 2 implies that $b^+_{\ell} = b^+_{\ell}$, and when $r < \bar{r}(\alpha_h) < \bar{r}(\alpha_h)$, we have $b^+_{\ell} > \beta_{\ell}$, which is a contradiction. Hence, there cannot be an equilibrium with $r < \bar{r}(\alpha_h)$.

Next, suppose there is an equilibrium with $r > \bar{r}(\alpha_h)$. As mentioned earlier, $r > \bar{r}(\alpha_h) \implies b^+_{\ell} < \beta_{\ell}$. From Lemma 2 we have $b^+_{\ell} \in [b^-_{\ell}, b^+_{\ell}]$, so in this case it must be that $b^+_{\ell} < \beta_{\ell}$. Now, the market-clearing constraint implies that $b^+_{\ell} > \beta_{\ell}$. Hence, as $\alpha_{\ell} \beta_{\ell} = \alpha_{h} \beta_{\ell}$, it must be that $\alpha_{\ell} b^+_{\ell} < \alpha_{h} b^-_{\ell}$. But then Lemma 2 implies that $b^+_{\ell} = b^-_{\ell}$, and when $r > \bar{r}(\alpha_h) > \bar{r}(\alpha_h)$, we have $b^+_{\ell} < \beta_{\ell}$, which is a contradiction. Hence, there cannot be an equilibrium with $r > \bar{r}(\alpha_h)$.

\textbf{Proof of Proposition 4}

There are only three possibilities in equilibrium:
(a) $\alpha_t b^*_t = \alpha_h b^*_h$, so that there are no net settlement transfers at date 2. In this case, as argued above, the market-clearing constraint implies that $b^*_t = \beta_h$ and $b^*_h = \beta_t$.

(b) $\alpha_t b^*_t > \alpha_h b^*_h$, so that bank $\ell$ must make a settlement transfer to bank $h$ at date 2. In this case, Lemma 2 implies that $b^*_t = b^-_t$ and $b^*_h = b^+_h$.

(c) $\alpha_t b^*_t < \alpha_h b^*_h$, so that bank $h$ must make a settlement transfer to bank $\ell$ at date 2. In this case, Lemma 2 implies that $b^*_t = b^+_t$ and $b^*_h = b^-_h$.

Next, suppose that $\alpha_h > \bar{\alpha}$. Then, as argued in part (i) above, we have $\bar{\alpha}(\alpha_h) > \bar{\alpha}(\alpha_h)$. We first rule out the possibility of an equilibrium with $r \geq \bar{\alpha}(\alpha_h)$ or $r \leq \bar{\alpha}(\alpha_h)$, and then show that an equilibrium exists for some $r \in (\bar{\alpha}(\alpha_h), \bar{\alpha}(\alpha_h))$.

Suppose that the equilibrium interest rate satisfies $r \geq \bar{\alpha}(\alpha_h)$ or $r \leq \bar{\alpha}(\alpha_h)$. Then, as argued in part (i) above, we have $b^*_h < \beta_h$. However, $b^*_t \in [b^-_t, b^+_t]$ in all cases by Lemma 2. Hence, there cannot be an equilibrium in which $b^*_t = \beta_h$, ruling out (a) above as a candidate for equilibrium.

Next, suppose that with $r \geq \bar{\alpha}(\alpha_h)$ there is an equilibrium satisfying case (b) above. Observe that $r > \bar{\alpha}(\alpha_h)$ implies that $r > 0$. Hence, from the definition of $b^*_t$ in equation (i), $b^-_t < b^*_h$, and as $r \geq \bar{\alpha}(\alpha)$ we have $b^-_h \leq \beta_t$. Therefore, $b^*_t = b^-_t < b^-_h \leq \beta_t$. Market-clearing now implies that we must have $b^*_h > \beta_h$. But then we must have $\alpha_t b^*_t < \alpha_h b^*_h$, which directly contradicts the assumption in case (b) that $\alpha_t b^*_t > \alpha_h b^*_h$.

Finally, as $b^-_h \leq \beta_t$ and $b^*_t < \beta_h$, the market-clearing constraint is immediately violated in case (c) above, so there cannot be such an equilibrium either. Hence, there cannot be an equilibrium with $r \geq \bar{\alpha}(\alpha_h)$.

Next, suppose that the equilibrium interest rate satisfies $r \leq \bar{\alpha}(\alpha_h)$ or $r \leq \bar{\alpha}(\alpha_h)$. Then, as argued in part (i) above, we have $b^*_h > \beta_h$, and $\beta_h > \beta_t$ by definition. However, $b^*_h \in [b^-_h, b^*_h]$ in all cases by Lemma 2. Hence, there cannot be an equilibrium in which $b^*_t = \beta_t$, ruling out (a) above as a candidate for equilibrium.

Next, suppose that with $r \leq \bar{\alpha}(\alpha_h)$ there is an equilibrium satisfying case (b) above. Observe that when $r > 0$, $b^*_t$ is increasing in $\alpha_t$. Hence, from the definition of $b^*_t$ in equation (i), we have $b^*_h > b^*_t$, and as $r \leq \bar{\alpha}(\alpha)$ we have $b^*_t \geq \beta_h$. Therefore, $b^*_h = b^*_t > b^-_h \geq \beta_t$. Market-clearing now implies that we must have $b^*_t < \beta_t$. But then we must have $\alpha_t b^*_t < \alpha_h b^*_h$, which directly contradicts the assumption in case (b) that $\alpha_t b^*_t > \alpha_h b^*_h$.

Finally, as $b^*_t > \beta_t$ and $b^*_t \geq \beta_h$, the market-clearing constraint is immediately violated in case (c) above, so there cannot be such an equilibrium either. Hence, there cannot be an equilibrium with $r \leq \bar{\alpha}(\alpha_h)$.

Therefore, if an equilibrium exists, it must have an equilibrium interest rate $r \in (\bar{\alpha}(\alpha_h), \bar{\alpha}(\alpha_h))$. At any such interest rate, as $r < \bar{\alpha}(\alpha_h)$, we have $b^-_h > \beta_t$. As $b^*_h \in [b^-_h, b^*_h]$, this rules out case (a)
as an equilibrium. Further, \( b^*_h \in [b^-_h, b^+_h] \) implies from market-clearing that \( b^*_\ell < \beta \), which violates the condition in case \((b)\) that \( \alpha b^*_\ell > \alpha b^*_h \). Therefore, any equilibrium must take the form in case \((c)\), with \( b^*_\ell = b^+_\ell \) and \( b^*_h = b^-_h \).

Now, observe that \( b^*_\ell \) and \( b^-_h \) are both continuous and strictly decreasing in \( r \) as long as \( \alpha \ell, \alpha h < 1 \). At the rate \( r = r(\alpha h) \), we have \( b^*_h = \beta \) and \( b^*_\ell > \beta \), so there is excess demand for funds. At the rate \( r = r(\alpha h) \), we have \( b^*_h = \beta \) and \( b^-_h < \beta \), so there is excess supply of funds. By continuity of \( b^-_h, b^*_\ell \), there must exist an \( r^* \in (r(\alpha h), r(\alpha h)) \) at which market-clearing is satisfied. Given that \( b^*_\ell + b^-_h \) is strictly decreasing in \( r \), there is only one such \( r^* \). Such an \( r^* \), with the associated values of \( b^-_h \) and \( b^*_\ell \) at that interest rate, constitutes a unique equilibrium when \( \alpha > \hat{\alpha} \).

**Proof of Proposition 5**

Consider \( \alpha_h \) approaching \( \hat{\alpha} \) from above. In the limit as \( \alpha_h \to \hat{\alpha} \), we have \( b^*_h = b^-_h = \beta \ell < b^*_\ell = b^*_\ell = \beta_h \). Further, \( b^-_h \) and \( b^*_\ell \) are continuous in \( \alpha_h \). Thus, for \( \alpha_h \) close to but strictly greater than \( \hat{\alpha} \), it must continue to be the case that \( b^*_h = b^-_h < b^*_\ell = b^*_\ell \).

Next, consider \( \alpha_h = 1 \). At this value, we have \( b^-_h = g(1 + \tau) \) and \( b^*_\ell = g(1 + \lambda r^*(1 - \alpha)) \). Now, when \( \alpha_h > \hat{\alpha} \), we have \( r^* > \hat{r} = f'(\hat{\beta}_h) - 1 \). Further, from Assumption \([\text{I}]\) part \((i)\), we have \( \tau < f'(\beta_h) - 1 \). Thus, \( \lambda r^*(1 - \alpha) > \tau \), so that when \( \alpha_h = 1 \), we have \( b^*_h = b^-_h > b^*_\ell = b^*_\ell \).

Now, \( \lambda r^*(1 - \alpha_h) > \tau \) implies that \( \lambda r^* > \tau \). Consider an increase in \( \alpha_h \). Keeping \( r^* \) fixed, this leads to a decrease in \( g(1 + r^* \lambda (1 - \alpha_h) + \tau \alpha_h) \), and hence to an increase in \( b^*_h \). The increase in \( \alpha_h \) has no effect on \( b^*_\ell \) when \( r^* \) is held fixed. Therefore, there is now excess demand for funds, which means that \( r^* \) must increase. Therefore, in the new equilibrium, \( r^* \) is higher, and hence \( b^*_\ell \) decreases, which means that \( b^*_h \) must increase.

As \( b^*_h \) is increasing in \( \alpha_h \) and \( b^*_\ell \) is decreasing in \( \alpha_h \), it follows that there exists some threshold value \( \hat{\alpha} \) such that \( b^-_h < b^*_\ell \) when \( \alpha_h < \hat{\alpha} \) and \( b^*_h > b^*_\ell \) when \( \alpha_h > \hat{\alpha} \).

**Proof of Proposition 6**

As in the proof of Proposition \(3\) we have \( z^*_\ell > z^*_h \) if and only if \( (1 - 2\alpha) b^*_\ell > (1 - 2\alpha_h) b^*_h \). Consider the following two cases:

(i) Suppose \( \alpha > \hat{\alpha} \) (so that we are in a payments equilibrium) \( f(x) = A \ln x \) where \( A > 0 \). In this case, when \( \tau > 0 \) we have \( b^*_h = \frac{A}{1 + r^* \lambda (1 - \alpha_h) + \tau \alpha_h} \) and \( b^*_\ell = \frac{A}{1 + r^* \lambda (1 - \alpha)} \). Therefore, the condition \( (1 - 2\alpha) b^*_\ell > (1 - 2\alpha_h) b^*_h \) holds if and only if \( \frac{1 - 2\alpha}{1 + r^* \lambda (1 - \alpha)} > \frac{1 - 2\alpha_h}{1 + r^* \lambda (1 - \alpha_h) + \tau \alpha_h} \). As \( r^* > 0 \) given Assumption \([\text{I}]\) part \((i)\), it follows (as in the proof of Proposition \(3\) part \((i)\)) that

\[
\frac{1 - 2\alpha}{1 + r^* \lambda (1 - \alpha)} > \frac{1 - 2\alpha_h}{1 + r^* \lambda (1 - \alpha_h) + \tau \alpha_h} \Rightarrow \frac{1 - 2\alpha}{1 + r^* \lambda (1 - \alpha)} > \frac{1 - 2\alpha_h}{1 + r^* \lambda (1 - \alpha_h) + \tau \alpha_h}.
\]

Thus, \( z^*_\ell > z^*_h \). Market-clearing now implies that \( z^*_\ell > 0 > z^*_h \).
Proof of Proposition 7
When \( \alpha_h > \bar{\alpha} \), in a market equilibrium, we have \( b_h^* = b_h^- = g(1 + r^*\lambda(1 - \alpha_h) + \tau \alpha_h) \), and \( b^*_\ell = b^+_\ell = g(1 + r^*\lambda(1 - \alpha_\ell)) \).

Suppose that \( \alpha_h > \bar{\alpha} \) and consider equilibrium values \( b_h^* \) and \( b^*_\ell \) at some \( \tau \). Consider a small decrease in \( \tau \), keeping \( r^* \) fixed. Then, \( 1 + r^*\lambda(1 - \alpha_h) + \tau \alpha_h \) decreases, so that \( b_h^- = g(1 + r^*\lambda(1 - \alpha_h) + \tau \alpha_h) \) increases. With \( r^* \) fixed, the change in \( \tau \) has no direct effect on \( b^*_\ell \). Therefore, if \( r^* \) is kept fixed, there is an excess demand for investable funds. Hence, \( r^* \) must increase in equilibrium.

The increase in \( r^* \) in turn implies an increase in \( b_h^- = b^+_\ell = g(1 + r^*\lambda(1 - \alpha_\ell)) \). Market-clearing now implies that \( b_h^* = b_h^- \) increases.

Proof of Proposition 8
Suppose that \( \tau < \bar{\tau} \) and \( \alpha > \bar{\alpha} \).

(i) From Proposition 4, in equilibrium we have \( b^*_h = b^-_h = g(1 + r^*\lambda(1 - \alpha_h) + \tau \alpha_h) \) and \( b^*_\ell = b^+_\ell = g(1 + r^*\lambda(1 - \alpha_\ell)) \). Therefore, \( \Gamma^* = |f'(b^*_h) - f'(b^*_\ell)| = |\tau \alpha_h - r^*\lambda(\alpha_h - \alpha_\ell)| \).

Define \( \psi = \tau \alpha_h - r^*\lambda(\alpha_h - \alpha_\ell) \). Observe that \( \frac{\partial \psi}{\partial \tau} = \alpha_h - \lambda(\alpha_h - \alpha_\ell) \frac{\partial r^*}{\partial \tau} \). Here, as shown in Proposition 7, in a payments equilibrium \( \frac{\partial r^*}{\partial \tau} < 0 \) (\( r \) falls as \( \tau \) increases) so it follows that \( \frac{\partial \psi}{\partial \tau} > 0 \).

Now, suppose \( \psi < 0 \) (i.e., suppose \( b^*_h > b^*_\ell \)). Then, a small increase in \( \tau \) leads to \( \psi \) becoming closer to zero, so that \( \Gamma^* \) falls. That is, a decrease in \( \tau \) leads to an increase in \( \Gamma^* \).

Next, suppose that \( \psi > 0 \) (i.e., suppose \( b^*_h > b^*_\ell \)). In this case, \( \psi \) rises as \( \tau \) increases. Thus, \( \Gamma^* \) increases in \( \tau \), and so falls when \( \tau \) falls.

Finally, if \( \psi = 0 \) (i.e., \( b^*_h = b^*_\ell \)), a reduction in \( \tau \) leads to a strict increase in \( \Gamma^* \).

(ii) From Proposition 4, when \( \tau \leq \bar{\tau} \), we have \( \Gamma^* = f'(b_h) - f'(b_\ell) = \tau(\alpha_h + \alpha_\ell) \). Thus, \( \frac{\partial \Gamma^*}{\partial \tau} = \alpha_h + \alpha_\ell > 0 \); i.e., \( \Gamma^* \) falls as \( \tau \) decreases.

Similarly, if \( \tau > \bar{\tau} \), we have \( b^*_h = \beta_\ell \) and \( b^*_\ell = \beta_h \). Both these quantities are independent of \( \tau \). Thus, it is immediate that a small decrease in \( \tau \) does not affect the inter-zonal productivity gap.

Proof of Proposition 9
(i) Suppose a proportion $\gamma$ of the population moves to CBDC. Then, $D$ reduces to $(1 - \gamma)D$, and $\lambda$ reduces to $(1 - \gamma)\lambda$. The right-hand side of the market-clearing constraint is therefore $\frac{2[(1 - \gamma)D + C]}{(1 - \gamma)\lambda}$.

As long as $\beta_{\gamma_h}, \beta_{\gamma_\ell}$ and $\tau$ satisfy Assumption 1, our previous analysis of equilibrium goes through.

Now, $b^*_{\gamma_\ell} = b^*_{\gamma_h} = g(1 + r^*(1 - \gamma)\lambda(1 - \alpha_\ell))$. Therefore,

$$\frac{db^*_{\gamma_\ell}}{d\gamma} = \frac{\partial b^*_{\gamma_\ell}}{\partial \gamma} + \frac{\partial b^*_{\gamma_\ell}}{\partial r} \frac{dr^*}{d\gamma} = -r^*\lambda(1 - \alpha_\ell)h'(b^*_{\gamma_\ell}) + (1 - \gamma)\lambda(1 - \alpha_\ell)h'(b^*_{\gamma_\ell}) \frac{dr^*}{d\gamma}$$

$$= \lambda(1 - \alpha_\ell)h'(b^*_{\gamma_\ell}) \{-r^* + (1 - \gamma) \frac{dr^*}{d\gamma} \}. \quad (23)$$

Similarly, $b^*_{\gamma_h} = b^*_{\gamma_h} = g(1 + r^*(1 - \gamma)\lambda(1 - \alpha_h) + \tau \alpha_h)$. Therefore,

$$\frac{db^*_{\gamma_h}}{d\gamma} = \lambda(1 - \alpha_h)h'(b^*_{\gamma_h}) \{-r^* + (1 - \gamma) \frac{dr^*}{d\gamma} \}. \quad (24)$$

Now, total differentiation of the market-clearing constraint w.r.t. $\gamma$ yields

$$\lambda \left\{-r^* + (1 - \gamma) \frac{dr^*}{d\gamma} \right\} \left[ (1 - \alpha_\ell)h'(b^*_{\gamma_\ell}) + (1 - \alpha_h)h'(b^*_{\gamma_h}) \right] = \frac{2\lambda R}{(1 - \gamma)^2} \frac{1}{\lambda^2}. \quad (25)$$

Now, when $R > 0$, the right-hand side is strictly positive. Further, $(1 - \alpha_\ell)h'(b^*_{\gamma_\ell}) + (1 - \alpha_h)h'(b^*_{\gamma_h}) < 0$. Hence, it must be that $-r^* + (1 - \gamma) \frac{dr^*}{d\gamma} < 0$. From the derivatives for $b^*_{\gamma_\ell}$ and $b^*_{\gamma_h}$, it now follows that each of these terms must increase as $\gamma$ increases.

(ii) The market-clearing constraint is $b^*_{\gamma_\ell} + b^*_{\gamma_h} = \frac{2(1 - \gamma)C + 2R}{(1 - \gamma)\lambda} = \frac{2C + \frac{2R}{\lambda}}{\lambda}$. By differentiating first with respect to $\gamma$ and then with respect to $C$, we obtain

$$\frac{d^2 b^*_{\gamma_\ell}}{d\gamma dR} + \frac{d^2 b^*_{\gamma_h}}{d\gamma dR} = \frac{2}{(1 - \gamma)^2} > 0. \quad (26)$$
References


